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INDEX.

ERRATA.

VOL. II.

Page 394, lines 13 and 14, *transpose* potential *and* projection.

VOL. III.

Page 3, line 27, *for* $\sum\limits_{k=0}$ *read* $\sum\limits_{k=0}^{k=n}$.

" 4, " 11, " " " "

" 10, " 29, " Fig. 6 *read* Fig. 4.

" 11, " 8, " 15 *read* 5.

" 12, *dele line* 20.

" 12, line 21, *insert* (*vi*) (x) *before* and.

" 12, " 22, *dele* (xii).

" 14, *dele lines* 3–5.

" 95, line 7, *before* χ *and* χ' *insert* 4π.

" 97, last line, " " " " "

" 158, line 24, *for* $m+n-z$ *read* $m+n-2$.

" 158, " 28, " xy " $-xy$.

" 159, " 3, " $\doteq -2$ " $=+2$.

" 160, " 6, " $J=C_{3,3}$ " $J=9\,C_{3,3}$.

" 160, " 8, " Δn^2 " $-\Delta u^2$

" 160, " 13, " $J=6\,hw$ " $8\,w=C_{6,3}$.

" 160, lines 6, 8, 19, 20, *for* h " 4.

" 160, " 23, 25, " h_i " L.

" 160, " 23, 26, " b " G.

" 160, line 27, *read* $=-4\,\{C_{2,2}^2\,(4\,EG-F^2)-C_{2,2}C_{6,2}(2\,LG$
$+\,2\,NE-MF)+C_{6,2}^2\,(4\,LN-M^2)\}$.

" 161, " 7, *for* $C_{3,3}$ *read* $C_{6,2}$.

" 162, " 11, " $\sqrt{\frac{2}{3}}$ " $0,\pm 1$.

" 162, " 19, " $\frac{1}{n}$ " $\frac{l}{n}$.

" 162, " 19, " l_{2m} " $l=m$.

" 185, " 1 of footnote, *for* $x^i.y^i.z^i\;xy_iz_i$ *read* $x_iy_iz_i$.

" 224, " 6, *for* c *read* e.

" 224, " 12, " $\log^{-n}, i=$ *read* $\log^{-n} i,=$.

Page 230, line 10, *insert* $-$ *before the second member.*
" 230, lines 12, 13, 14, *for* X_0 *read* χ_0.
" 231, " 6, 12, *change sign of the integral.*
" 232, " 4, 8, " " " "
" 232, " 20, 21, 22, *for* $-$ *read* $+$.
" 233, " 7, 8, 9, *dele terms containing* ξ_{n+1}.
" 233, " 12, *change sign of the integral.*
" 234, " 1, 3, " " " "
" 234, line 11, *change sign of second and third members.*
" 234, lines 13, 14, 15, *for first sign* $+$ *read* $-$.
" 235, line 20, *change sign of second member.*
" 236, " 12, *insert factor* $(2s-1)$ *before the integral.*
" 253, lines 25, 27, *change sign of second member.*
" 253, line 29, *insert* $-$ *before second member of each equation.*
" 255, " 1, *dele* surely *and* no.
" 260, " 8, *for* S *read* ρ.
" 265, " 31, " M " M_1.
" 267, " 2, *insert* $\frac{1}{r}$ *under the integral sign.*
" 267, " 9, *dele* In this case.
" 267, " 18, *for* L_0, M_0, N_0 *read* R_0, S_0, T_0 *respectively.*
" 267, lines 21, 23, 26, interpret $\left(\frac{dM_0}{d\nu}\right)^2$ in a quaternion sense, or replace it by $\left(\frac{dF_0}{d\alpha}\right)^2+\left(\frac{dG_0}{d\beta}\right)^2+\left(\frac{dH_0}{d\gamma}\right)^2$, where α, β, γ, are lines in the direction of greatest increase of F_0, G_0, H_0, respectively, or by $\frac{1}{2}\Delta^2(M_0^2)+F_0F_2+G_0G_2+H_0H_2$.
" 267, lines 21, 23, 26, *for the coefficient* 2 *read* 4.
" 345, line 17, *for* $\phi(x+h)$ *read* $\phi(x+mh)$.
" 346, *the second set of equations* (*a*) (*b*) (*c*) (*d*) *should be designated* (*a*)′ (*b*)′ (*c*)′ (*d*)′.
" 348, line 2 of equation (16), *for* $\left(\frac{d\theta_1}{dh}\right)^2$ *read* $\left(\frac{d\theta_1}{dh}\right)^2_{h=0}$.
" 351, " 14, *for* $\left(\frac{d^2\theta_1}{dh^2}\right)_{h=0}$ *read* $\left(\frac{d^2\theta}{dh^2}\right)_{h=0}$.
" 354, " 18, *insert* $+$ *between* $\frac{d\theta_1}{dh}$ *and* $\frac{2d\theta_2}{dh}$.
" 354, " 20, *for* $\frac{1}{16}$ *read* $\frac{1}{64}$.

Regular Figures in n-dimensional Space.

By W. I. Stringham,

Fellow of the Johns Hopkins University.

A pencil of lines, diverging from a common vertex in n-dimensional space, forms the edges of an n-fold (short for n-dimensional) angle. There must be at least n of them, for otherwise they would lie in a space of less than n dimensions. If there be just n of them, combined two and two they form 2-fold face boundaries; three and three, they form 3-fold trihedral boundaries, and so on. So that the simplest n-fold angle is bounded by n edges, $\frac{n(n-1)}{2}$ faces, $\frac{n(n-1)(n-2)}{1.2.3}$ 3-folds, in fact, by $\frac{n!}{k!(n-k)!}$ k-folds. Let such an angle be called *elementary*. Fig. 1 represents the symmetrical arrangement, in three-dimensional perspective, of the four trihedral boundaries of an elementary 4-fold angle. When put into space of four dimensions, the faces of the tetrahedra, which lie adjacent to the common vertex, are to be brought into coincidence two and two; the edges will then fall together three and three.

A regular angle is defined as one all of whose boundaries of any given number of dimensions are the same in form and magnitude. The number of regular n-fold angles that can be formed out of sets of regular $(n-1)$-fold angles is limited by the number of those symmetrical arrangements of the $(n-1)$-fold angles about a vertex in an $(n-1)$-fold space, which involve the symmetrical and equal distribution of all of their boundaries. To give concreteness to the idea, let equal distances be measured from the vertex of a regular n-fold angle on its edge boundaries. These equal lines must terminate in the summits of a regular $(n-1)$-fold figure; from which it follows that the different kinds of regular n-fold angles are just equal in number to the regular $(n-1)$-fold figures or, what is the same thing, to the regular distributions of points on the $(n-1)$-fold sphere. Hence there suggests itself an obvious criterion for the possibility of any regular n-fold angle, viz.:

(A.) *The number of its edge boundaries shall be equal to the number of summits of some regular $(n-1)$-fold figure.*

Conformably with a notation hereafter employed in connection with the regular figures (see p. 8), this criterion may be expressed analytically in the form

$$(n)_1^0 = (n-1)_0^{n-1}, \quad \text{or} \quad (n)_1^0 = (N-1)_0,$$

where $(N-1)_0 = (n-1)_0^{n-1}$ is the number of summits of some regular $(n-1)$-fold figure, and is to have all possible values under the conditions named, and $(n)_1^0$ is the number of edges which are to enter into the formation of the regular n-fold angle; $(n)_1^0$ can have only such values as are equal to the different possible values of $(n-1)_0^{n-1}$. The criterion is at least sufficient, and in this discussion I shall assume it to be necessary, reserving the question for future consideration, if such be required.

In particular, the number of regular 4-fold angles that can be made up of regular polyhedral angles is five. Out of trihedral angles three regular 4-fold angles can be formed, corresponding to the three regular polyhedra which have triangular boundaries. Similarly there is one regular distribution for tetrahedral and one for pentahedral angles. The former consists of a set of six angles corresponding to the faces of the cube, and the latter of twelve angles corresponding to the faces of the dodekahedron.

Three of the five regular 4-fold angles are shown in Plates I., II., laid out in three-dimensional perspective. Figs. 1 and 3 belong to what may be called the tetrahedroidal, Fig. 5 to the oktahedroidal, and Fig. 7 to the hexahedroidal 4-fold angles. The other two, not represented in the plates, are dodekahedroidal and ikosahedroidal.

A *complete* n-fold figure is defined as one which is limited on every hand by complete $(n-1)$-fold figures which are themselves limited by complete $(n-2)$-fold figures, and so on. In other words, a complete figure has no gaps in it. It will be convenient to designate as an n-fold *polyhedroid* the n-dimensional figure which is bounded by $(n-1)$-fold flat (not curved) figures; and in particular to call the figure which has m $(n-1)$-fold boundaries an n-fold (m)-hedroid; thus, an oktahedron is a 3-fold (8)-hedroid, or oktahedroid.

An n-fold *elementary* pyramid is the simplest n-fold figure. It has $n+1$ summits, $\frac{(n+1)\,n}{2}$ edges, $\frac{(n+1)\,n\,(n-1)}{1.2.3}$ faces, and in general it has $\frac{(n+1)!}{k!\,(n-k+1)!}$ $(k-1)$-dimensional boundaries. Thus the successive terms of the left-hand member of

$$(1-1)^{n+1} = 0, \tag{1}$$

beginning with the second and ending with the $(n+1)^{\text{th}}$, represent the numbers

of boundaries of the figure, the second term standing for summits, the third for edges, the fourth for two-dimensional boundaries, and so on. This is, in fact, a generalized form, for this special class of n-fold figures, of Euler's equation for polyhedra. The summits of a *regular* elementary pyramid are all equidistant from each other, and may be said to have *absolute* symmetry of position with reference to each other. The elementary pyramids are self-reciprocal in the sense in which a tetrahedron is said to be self-reciprocal; that is to say, if the centres of the $(n-1)$-fold boundaries be taken for the summits of a new figure, then the old summits will be replaced by an equal number of $(n-1)$-fold boundaries, and in general the same reciprocal relation will exist between any two sets of boundaries which are equidistant from the two ends of the series; viz. a set of k-fold and a set of $(n-k-1)$-fold boundaries.

In particular, the 4-fold pentahedroid has 5 summits, 10 edges, 10 triangular and 5 tetrahedral boundaries. To construct this figure select any one summit of each of four tetrahedra and unite them. Bring the faces, which lie adjacent to each other, two and two into coincidence. There will remain four faces still free; take a fifth tetrahedron, and join each one of its faces to one of these four remaining ones. The resulting figure will be the complete 4-fold pentahedroid.

It has been seen that Equation (1) is a particular case of the generalized Eulerian polyhedral formula. Before proceeding with the discussion of the other regular figures, I propose to give a demonstration of the generalized formula for n-fold figures in general. Let N_k represent the number of k-fold boundaries to an n-fold figure; that is, say it has N_0 summits, N_1 edges, N_2 faces, etc. Conformably with this notation, Equation (1) will assume the form

$$\begin{aligned}(1-1)^{n+1} &= 1-(n+1)+\frac{(n+1)\,n}{1.2}-\cdots\pm(n+1)\mp 1\\ &= 1-N_0+N_1-\cdots\pm N_{n-1}\mp N_n = 1-\sum_{k=0}(-)^k N_k = 0, \quad (a)\end{aligned}$$

wherein N_n is to be counted as 1 when the figure is simple, as here, i. e. not an assemblage of two or more n-fold figures. It is to be shown that this equation is true for any n-fold figure whatever, simple or complex; — by a *complex* figure I mean one made up of two or more simple figures joined together. It is important to observe that, when the figure is complex, it may either be treated as such, in which case N_n is the number of figures in the group, or it may be regarded as an $(n+1)$-fold simple figure, in which case another unit is to be added to N_n, viz. some selected n-fold outline of the figure, not counted before, is to be counted as 1 in N_n. Also a term N_{n+1}, counting as 1 for the $(n+1)$-fold

figure, must be added to the formula. The figure may be actually converted into a simple $(n+1)$-fold one by properly distorting it in the $(n+1)$-dimensional space. The formula thus remains the same in form for the $(n+1)$-fold simple figure as for the n-fold complex one, the upper limit of k for the former case being $n+1$, for the latter n. The n-fold complex figure is, in fact, simply the projection of the $(n+1)$-fold one into one of its n-fold boundaries. On this principle is based the following very simple proof that the function (α) must vanish for any n-fold simple figure, or what is the same thing, for any $(n-1)$-fold complex one.

For brevity, write the function (α), corresponding to a simple n-fold figure of arbitrary construction, in the form

$$\Phi(n) = 1 - \sum_{k=0} (-)^k N_k. \tag{α}$$

The object of the proof is to show that $\Phi(n) = 0$. For the purposes of the proof, the boundaries of the figure may be supposed to be elastic, so as to admit of being distorted without incurring any change in their numerical relations. First, project the figure as a whole into one of its $(n-1)$-fold boundaries; the term $N_n = \pm 1$ disappears, together with a unit, of opposite sign from N_n, out of the term N_{n-1}; these units cancel each other, and the function becomes $\Phi(n-1)$. The figure is now $(n-1)$-dimensional and complex, and N_{n-1} is the number of its $(n-1)$-fold constituents. Taking any one of these constituents, project it into one of its own $(n-2)$-fold boundaries, distorting the other boundaries of the figure sufficiently to allow of this being done; taking care, at the same time, that no two boundaries shall fall into coincidence in the projected figure, and so one of them disappear. In this process the $(n-1)$-fold figure is obliterated and also an $(n-2)$-fold one; viz., the former has collapsed and no longer exists, while the latter is replaced by the other $(n-2)$-fold figures which have been projected into it. Thus, a unit is extracted from each of the terms N_{n-1}, N_{n-2}, and the two units cancel each other in consequence of N_{n-1} and N_{n-2} having opposite signs. This operation, repeated upon all save one of the other constituent $(n-1)$-fold parts of the complex $(n-1)$-fold figure, will reduce the figure to a simple $(n-1)$-fold one, for which N_{n-1} is equal to 1. Observe that at each step of this reduction *one* unit and *only one* is extracted from each of the terms N_{n-1}, N_{n-2}. The projection can always be made so as to insure this result.*

* It is of no consequence if some of the boundaries intersect each other in the projected figure. But even this may be avoided by a proper order in making the projections and by a proper distortion of the figure. Thus, if two $(n-1)$-fold boundaries happen to lie in the same $(n-1)$-dimensional flat space, such a crossing of boundaries will take place; but the two $(n-1)$-fold boundaries may, by a slight distortion of the figure, be made to lie in different $(n-1)$-dimensional spaces.

This simple $(n-1)$-fold figure can now be projected as a whole into one of its $(n-2)$-fold boundaries, thus becoming an $(n-2)$-fold complex figure; N_{n-1} will disappear entirely from the formula, which thus becomes $\Phi(n-2)$. By a repetition of the foregoing process, the number of dimensions of space to which the figure belongs is diminished by successive units, simultaneously with which the upper limit of k in the Φ function is also diminished by successive units. The outcome of this is that in the limit, on arriving at zero-dimensional space, the function reduces to the simple identity $\Phi(0) = 1 - 1 = 0$. Now since the quantities which were extracted from the terms of $\Phi(n)$ have cancelled each other in pairs, and the function is zero in the limit, therefore, in general, $\Phi(n) = 0$, which was to be proved.

I pass to the further consideration of the regular figures, and first of those belonging to four-dimensional space. The 4-fold pentahedroid has already been described. The next simplest is the 4-fold orthogonal figure; it is an oktahedroid. It may be generated by giving the 3-fold cube a motion of translation in the fourth dimension in a direction perpendicular to the three-dimensional space in which it is situated. Each summit generates an edge, each edge a square, each square a cube. To the numbers of boundaries thus generated must be added twice the numbers of summits, edges, squares, and cubes of the original figure (viz. each summit, edge, square, and cube must be counted once for its initial and once for its final position). Using D_0, D_1, D_2, D_3, D_4 in place of the $N_0, N_1, \ldots\ldots$ to represent in particular the numbers of the boundaries of 4-fold figures, we have for the case in hand

$$\begin{aligned}
D_0 &= \quad\;\;\; 8 + \;\, 8 = 16 = 2^4 \\
D_1 &= 12 + \;\, 8 + 12 = 32 = 4 \,.\, 2^3 \\
D_2 &= \;\, 6 + 12 + \;\, 6 = 24 = \frac{4\,.\,3}{2}\, 2^2 \\
D_3 &= \;\, 1 + \;\, 6 + \;\, 1 = \;\, 8 = \frac{4\,.\,3\,.\,2}{1\,.\,2\,.\,3}\, 2 \\
D_4 &= \qquad\qquad\qquad\;\; = 1.
\end{aligned}$$

The equation (α) for this case becomes

$$1 - D_0 + D_1 - D_2 + D_3 - D_4 = 1 - (2-1)^4 = 0.$$

By an extension of the above process of construction to higher dimensional figures it is easy to show that in general the boundaries of the n-fold orthogonal figure give the relation

$$1 - N_0 + N_1 - N_2 + N_3 - \ldots\ldots \mp N_n = 1 - (2-1)^n, \tag{2}$$

the upper sign corresponding to n even, the lower to n odd.

The 4-fold hexadekahedroid and its analogues in n-fold space are the reciprocals of the corresponding orthogonal figures, and the Φ function for these is (2) written backwards, i. e.

$$1 - N_0 + N_1 - N_2 + N_3 - \cdots \mp N_n = (1-2)^n \mp 1 = 0. \qquad (3)$$

But it will be interesting to see how the 4-fold figure may be completely determined by purely geometrical considerations. Take the eight middle points of the cube boundaries of the 4-fold octahedroid as summits; $D_0 = 8 = 4 \,.\, 2$. These points are the eight extremities of the four mutually perpendicular diameters of the 4-fold sphere. The edges of the figure are found by joining each summit with each of the other summits except its antipode, i. e. with six adjacent ones; so that the number of edges diverging from a vertex is six. Hence $D_1 = \frac{6 D_0}{2} = \frac{4\,.\,3}{1\,.\,2}\, 2^2$. Consider the four points, equidistant from each other, which lie at the extremities of four mutually perpendicular radii of the 4-fold sphere. They have absolute symmetry of position in three-dimensional space, and so are the summits of a regular tetrahedron. Any fifth point of the series is an antipode of one of the four already selected, and such a group of five points cannot lie in a space of three dimensions, so that four of the eight points only can be taken as constituting the summits of the three-dimensional boundaries; hence these boundaries are tetrahedra. Their number is found by enumerating the groups of four out of the eight points, excluding groups which contain antipodes; so that $D_3 = 1 + 4 + \frac{4\,.\,3}{2} + \frac{4\,.\,3\,.\,2}{1\,.\,2\,.\,3} + 1 = 16 = 2^4$; whence, also, $D_2 = \frac{4 D_3}{2} = 32 = \frac{4\,.\,3\,.\,2}{1\,.\,2\,.\,3}\, 2^3$. Therefore D_0, D_1, D_2, D_3, D_4 do satisfy the relation (3), as was predicted.

It appears, then, that every space has at least three regular figures. I shall hereafter refer to these as the binomial tri-group in n-dimensional space. The three figures in question are, in the notation we have adopted, the n-fold $(n+1)$-hedroid, the n-fold $(2\,n)$-hedroid, and the n-fold (2^n)-hedroid.

Figs. 1, 3, 5 represent respectively the summits, one in each figure, of the 4-fold pentahedroid, oktahedroid, and hexadekahedroid, with the 3-fold boundaries of the summit spread out symmetrically in three-dimensional space. Figs. 2, 4, 6 represent the complete projections of these three regular figures upon a plane. The tetrahedral boundaries of the pentahedroid (Fig. 2) are *a b c d*, *b c d e*, *c d e a*, *d e a b*, *a b c e*. The eight cube boundaries of the oktahedroid (Fig. 4) are *a b c d e f g h*, *l m n p q r s t*, *a b c d l m n p*, *e f g h q r s t*, *a c l n e g q s*, *b d m p f h r t*, *n c d p g h s t*, *a b l m e f q r*. The sixteen tetrahedral boundaries of the hexadekahedroid (Fig. 6) are *a b c d*, *a′b′c′d′*, *a b′c′d′*, *a′b c d*, *a b′c d*, *a′b c′d′*,

$a\,b\,c'\,d$, $a'\,b'\,c\,d'$, $a\,b\,c\,d'$, $a'\,b'\,c'\,d$, $a\,b\,c'\,d'$, $a'\,b'\,c\,d$, $a'\,b\,c'\,d$, $a\,b'\,c\,d'$, $a'\,b\,c\,d'$, $a\,b'\,c'\,d$.
The accented letters are the antipodes of the unaccented ones.

Now in building up all possible 4-fold regular figures we have five regular 3-fold figures to deal with. Let us take successively these five figures, and using each as a framework upon which to arrange symmetrically the 3-fold boundaries to one of the summits of the 4-fold figures, consider all the possible cases. (The framework above referred to I shall call the *frame* figure, and the $(n-1)$-fold figure, out of a group of which the n-fold one is to be constructed, the *generating* figure.) The arrangements, with the 3-fold figures, which give regular distributions amongst the summits of the frame figure, i. e. those which satisfy the criterion (A) of page 2, for a regular angle, are —

(*i.*)	4 tetrahedra	upon a	tetrahedral	frame.
(*ii.*)	4 hexahedra	"	"	"
(*iii.*)	4 dodekahedra	"	"	"
(*iv.*)	8 tetrahedra	upon an	octahedral	frame.
(*v.*)	8 hexahedra	"	"	"
(*vi.*)	8 dodekahedra	"	"	"
(*vii.*)	20 tetrahedra	"	ikosahedral	"
(*viii.*)	20 hexahedra	"	"	"
(*ix.*)	20 dodekahedra	"	"	"
(*x.*)	6 octahedra	upon a	hexahedral	"
(*xi.*)	12 ikosahedra	"	dodekahedral	"

These eleven cases naturally fall into reciprocal pairs wherein the generating figure of the one case is the reciprocal of the frame figure of the other, and into self-reciprocal single categories in which the generating and frame figures are the reciprocals of each other. The reciprocal pairs are (*ii*) (*iv*), (*iii*) (*vii*), (*vi*) (*viii*), and the self-reciprocal categories are (*i*), (*v*), (*ix*), (*x*), (*xi*). The resulting figures will group themselves into three classes, — real, imaginary, and infinite. The imaginary figures are those which cannot be built up in consequence of their enclosing too much space; the infinite ones are those which completely fill up — saturate, so to speak — infinite three-dimensional space.

For the rejection of those cases which give rise to imaginary figures and the complete determination of the real figures, the following general criteria, applicable to an n-fold regular figure, will be useful, and are obviously necessary: —

(B.) *The number of* (n − 1)*-fold boundaries which border upon any summit of a regular* n*-fold figure must be less than the number of those* (n − 1)*-fold boundaries which can be joined together — with their* (n − 2)*-fold boundaries coincident two and two — about a point in* (n − 1)*-dimensional space.*

(C.) *In a regular* n-*fold figure, every boundary of a given number of dimensions must have lying adjacent to it the same number of equal boundaries of any other number of dimensions.*

Analytically expressed, the criterion (C) is equivalent to

$$(n)_r^s N_s = (n)_s^r N_r, \tag{4}$$

where N_s, N_r represent the number of s- and r-fold boundaries to the regular figure, and $(n)_r^s =$ the number of r-fold boundaries to an s-fold boundary, $(n)_s^r$ = the number of s-fold boundaries to an r-fold one. Here n is the number of dimensions of the space to which the figure belongs, and r, s may have any values from 0 to n inclusive. It is obvious that $(n)_k^n = N_k$.

For the application of the criterion (B) to the 4-fold figures, the following table of lengths of edges of the five regular solids, each inscribed in a sphere whose radius is unity, will be useful: —

Tetrahedron,	$l = 1.632994$,
Hexahedron,	$l = 1.154700$,
Oktahedron,	$l = 1.414214$,
Dodekahedron,	$l = \ .713644$,
Ikosahedron,	$l = 1.051462$.

Case (v) gives the only infinite figure in the above scheme. Its construction is obvious.

Cases (i), (ii), (iv) give real figures, viz. the pentahedroid, the oktahedroid, and the hexadekahedroid.

Cases (vi), (ix). Let the three summits of a dodekahedron, which are nearest to and equidistant from another of its summits S, be placed in coincidence with three mutually adjacent summits of a regular oktahedron inscribed in the unit sphere, and let the summit S be directed inward towards the centre of the sphere. The length of an edge of the oktahedron being 1.414214, the calculated length of edge of the dodekahedron is .874048, or less than the radius of the sphere. Hence the summit S will fall short of the centre of the sphere; wherefore eight trihedral angles of regular dodekahedra more than fill up the three-dimensional space about a point. Case (vi), therefore, gives rise to an imaginary figure. Case (ix) likewise gives an imaginary figure, since l is less for the ikosahedron than for the oktahedron.

Case ($viii$). This, being the reciprocal of Case (vi), must give an imaginary figure. It is otherwise seen to be imaginary from the fact that the length of edge of the frame figure (the ikosahedron) being 1.051462, that of the hexahedron is $1.051462 \div \sqrt{2} = .743496$, or less than the radius of the sphere.

Case (xi). Five summits of an ikosahedron adjacent to another summit S are to be placed in coincidence with the five summits of a pentagonal face of the frame dodekahedron, with S inside the sphere. The length of edge of the dodekahedron being .713644, that of the ikosahedron is the same, or less than the radius of the sphere. Hence the figure is imaginary.

Case (x). Place the four summits of an oktahedron (Fig. 6), which lie at the corners of a square section, in coincidence with the four corners of one of the faces of the inscribed cube. The edge of the cube is 1.1547, greater than the radius of the sphere. Hence this case gives rise to a real figure.

Case (iii). The length of an edge of the frame figure (the tetrahedron) being 1.632994, the calculated length of an edge of the dodekahedron is 1.009261, or greater than the radius of the sphere. The figure is real.

Case (vii). The length of edge of the frame figure (the ikosahedron) is 1.051462, and that of the tetrahedron is the same, or greater than the radius of the sphere. The figure is real.

There are thus three 4-fold figures yet to be determined, the self-reciprocal figure of Case (x) and the reciprocal group of Cases (iii), (vii).

That the figure of Case (x) is self-reciprocal is shown by the relations

$$(d)_0^3\, D_3 = (d)_3^0\, D_0\,,\ (d)_1^2\, D_2 = (d)_2^1\, D_1\,,$$

special forms of the equation of Criterion (C), wherein for the case in hand

$$(d)_1^2 = (d)_2^1 = 3\,,\ (d)_0^3 = (d)_3^0 = 6.$$

The figure is a 4-fold ikosatetrahedroid. A summit out of each of six oktahedra joined together make up one summit of the new figure. Fig. 7 shows such a summit with six oktahedral boundaries arranged about it symmetrically in three-dimensional space. Conceive Fig. 7 to be transported into four-dimensional space and the interstices between the adjacent triangular faces to be closed up by joining those faces two and two; the figure assumes a form whose projection is represented in Fig. 8 with dotted lines omitted. Adjust to this figure twelve other oktahedra in a symmetrical manner; three of these oktahedra are represented by the dotted lines of Fig. 8. Again, close up the interstices between the adjacent faces; the outline of the figure assumes a form whose projection is represented in Fig. 9. Now, conceive this figure to be turned inside out. There will be left in the middle of the figure a vacant space of exactly the form of Fig. 8 with the dotted lines omitted; such a group of six oktahedra is therefore required to complete the 4-fold figure. By counting up all the constituent oktahedral summits and other boundaries the reader may satisfy himself that the summits of the 4-fold figure are filled to saturation, and that the figure is in other respects complete and regular.

The number of oktahedral boundaries is $D_3 = 6 + 12 + 6 = 24$; of summits, $D_0 = \frac{6 \cdot 24}{6} = 24$; of triangular faces, $D_2 = \frac{8 \cdot 24}{2} = 96$; of edges, $D_1 = \frac{8 \cdot 24}{2} = 96$. Thus the equation $\Phi(n)$ for this case is

$$1 - D_0 + D_1 - D_2 + D_3 - D_4 = 1 - 24 + 96 - 96 + 24 - 1 = 0. \quad (5)$$

This figure bears peculiar relations to the oktahedroid and the hexadekahedroid. It is easily shown that the edge of the oktahedroid is equal to the radius of the circumscribed 4-fold sphere; in fact, $4\, l^2 = (2\, r)^2$, or $l = r$, where r is the radius of the sphere and l is the length of edge of the oktahedroid. Now, the semidiagonal of the cube whose edge is r, is $r\, \frac{1}{2}\sqrt{3}$; therefore, the distance from the centre of the 4-fold sphere to the centre of one of the cube boundaries of the inscribed regular oktahedroid is $\sqrt{r^2 - r^2(\frac{1}{2}\sqrt{3})^2} = \frac{1}{2}\, r$. Hence the point of intersection, P, of the sphere with the radius drawn through the centre of one of these cubes is at a distance r from each of the summits of the cube; in other words, the eight lines joining this point with the summits of the cube are in length equal to an edge of the cube, therefore also to the radius of the sphere. These lines, in fact, are edges of a regular ikosatetrahedroid. There are $8 \times 8 = 64$ of the lines joining the points P with the summits of the oktahedroid, and these together with the 32 edges of the oktahedroid constitute the 96 edges of the ikosatetrahedroid. Moreover, the summits of the latter figure are the 16 summits of the oktahedroid plus the 8 points P. Again, each of the 32 edges of the oktahedroid has 3 cubes bounding it; hence, there are 3 triangular faces of the ikosatetrahedroid corresponding to each of these edges, or in all $3 \times 32 = 96$. Finally, the 24 2-fold faces of the oktahedroid are 24 square plane sections of the oktahedral boundaries of the ikosatetrahedroid. Eight of the summits of the ikosatetrahedroid constitute the summits of a regular hexadekahedroid; but the edges and 2-fold boundaries of the latter figure are quite distinct from those of the former.

A projection of the ikosatetrahedroid may be constructed by drawing all the diagonal lines of the 8 cubes of Fig. 6, and regarding the half-diagonals as edges of the new figure and the middle points of diagonals as summits.

Having determined the figure of either Case (*iii*) or (*vii*), that of the other will be determined as its reciprocal. The process of building up the figure of Case (*vii*) is graphically illustrated in Figs. 10–17. Twenty tetrahedral summits are required to make up a summit of the 4-fold figure. By joining together a single group of 20 tetrahedra and bringing the adjacent faces two and two into coincidence, a figure is produced whose projection in three-dimensional space has the

form of Fig. 10.* In Fig. 11, 20 more tetrahedra have been added; the addition of 20 more to Fig. 11 gives Fig. 12; the addition of $5 \times 12 = 60$ to Fig. 12 gives Fig. 13; the addition of $2 \times 30 = 60$ to Fig. 13 gives Fig. 14; the addition of $5 \times 12 = 60$ to Fig. 14 gives Fig. 15. The manner of distributing the tetrahedra as they are added is obvious. This last figure contains 20 indentations into which trihedral, and 12 into which pentahedral angles must be inserted. Form 20 groups of 2 tetrahedra each, as represented in Fig. 16, having trihedral angles at R, R'; and form 12 groups of 15 tetrahedra each, as represented in Fig. 17, having pentahedral angles at S, S'. These $20 + 12 = 32$ groups, together with two of the groups of Fig. 15, will fit into each other and complete the regular 4-fold figure. That the summits of this figure are all saturated can be verified by actual count. Thus the number of tetrahedral boundaries to the regular figure is $2\,(20 + 20 + 30 + 60 + 60 + 60) + 2 \times 20 + 5 \times 12 = 600$. Applying the criterion (C) in order to determine the other boundaries, we have

$$(d)_3^0 = 20, \quad (d)_0^3 = 4, \quad (d)_3^1 = 5, \quad (d)_1^3 = 6, \quad (d)_3^2 = 2, \quad (d)_2^3 = 4;$$

whence, in consequence of the relation $(d)_r^s\, D_s = (d)_s^r\, D_r$,

$$\begin{aligned} 20\, D_0 &= 4 \times 600, & D_0 &= 120; \\ 5\, D_1 &= 6 \times 600, & D_1 &= 720; \\ 2\, D_2 &= 4 \times 600, & D_2 &= 1200; \end{aligned}$$

so that $\Phi(n)$, for this case, is

$$1 - D_0 + D_1 - D_2 + D_3 - D_4 = 1 - 120 + 720 - 1200 + 600 - 1 = 0. \quad (6)$$

In the nomenclature we have adopted the figure is a 4-fold (600)-hedroid, or hexakosioihedroid.

The reciprocal of the figure last described is evidently a 4-fold (120)-hedroid, or hekatonikosihedroid. This figure may be constructed as follows: Each summit must have 4 constituent dodekahedral summits. Take a single dodekahedron as a foundation upon which to build. Surround it with 12 other dodekahedra, as represented in Fig. 18. The successive additions to this figure of 20, 12, and 30 dodekahedra produce respectively the figures 19, 20, and 21. Now 19 and 20 are the negatives of each other in the photographic sense, and will fit together so as to complete the regular figure. Thus the number of dodekahedral boundaries to the figure is $2\,(1 + 12 + 20 + 12) + 30 = 120$. Applying the criterion (C) we have

* The constituent tetrahedra represented in the projected figures are not regular in so far as they have not equal regular faces. But that is evidently of no consequence, so long as the interstices are closed in the proper way and the right faces are left free to admit of further additions. In other respects the projected figures may be distorted *ad libitum*. Figs. 11, 13 are the two regular star-faced dodekahedra of Poinsot.

$$(d)^0_3 = 4, \quad (d)^3_0 = 20, \quad (d)^1_3 = 3, \quad (d)^3_1 = 30, \quad (d)^2_3 = 2, \quad (d)^3_2 = 12\,;$$

$$\begin{aligned} 4\,D_0 &= 20 \times 120, & D_0 &= 600\,; \\ 3\,D_1 &= 30 \times 120, & D_1 &= 1200\,; \\ 2\,D_2 &= 12 \times 120, & D_2 &= 720\,; \end{aligned}$$

$$1 - D_0 + D_1 - D_2 + D_3 - D_4 = 1 - 600 + 1200 - 720 + 120 - 1 = 0, \qquad (7)$$

which last is (6) written backwards, as it should be.

Passing to the discussion of the regular 5-fold figures, the several arrangements to be considered are

(*i.*)	5	(5)-hedroids	upon a	(5)-hedroidal	frame.
(*ii.*)	5	(8)-hedroids	" "	"	"
(*iii.*)	5	(120)-hedroids	" "	"	"
(*iv.*)	16	(5)-hedroids	" "	(16)-hedroidal	"
(*v.*)	16	(8)-hedroids	" "	"	"
(*vi.*)	16	(120)-hedroids	" "	"	"
(*vii.*)	8	(24)-hedroids	" "	(8)-hedroidal	"
(*viii.*)	24	(16)-hedroids	" "	(24)-hedroidal	"
(*ix.*)	600	(5)-hedroids	" "	(600)-hedroidal	"
(*x.*)	600	(8)-hedroids	" "	"	"
(*xi.*)	600	(120)-hedroids	" "	"	"
(*xii.*)	120	(600)-hedroids	" "	(120)-hedroidal	"

The reciprocal pairs in this scheme are (*ii*) (*iv*), (*vii*) (*viii*), (*iii*) (*ix*), and the cases giving self-reciprocal figures are (*i*), (*v*), (*xi*), (*xii*).

For present purposes the following approximate lengths of edges of the 4-fold regular figures inscribed in the unit 4-fold sphere are sufficiently accurate:

4-fold	(5)-hedroid,	$l = 1.58113$,
"	(8)-hedroid,	$l = 1.00000$,
"	(16)-hedroid,	$l = 1.41421$,
"	(24)-hedroid,	$l = 1.00000$,
"	(120)-hedroid,	$l < 1$,
"	(600)-hedroid,	$l < 1$.*

* In general, the length of edge of the n-fold $(n+1)$-hedroid, $(2\,n)$-hedroid and (2^n)-hedroid are respectively

$$r_n \sqrt{\frac{2\,(n+1)}{n}}, \quad r_n \frac{2}{\sqrt{n}}, \quad r_n \sqrt{2}\,,$$

where r_n is the radius of the circumscribed n-fold sphere. The first expression is a consequence of the fact that the distance of the centre of the n-fold $(n+1)$-hedroid to one of its summits is $\frac{n}{n+1}$ times the distance from a summit to the centre of the opposite $(n-1)$-fold boundary. The second expression is obtained from the equation $(2\,r_n)^2 = n l_n^2$, where l_n is the length of edge, and the third is the quadrant of a great circle on the n-fold sphere. Observe that in an infinite-dimensional space the second expression is zero and the first and third are iden-

Cases (*i*), (*ii*), (*iv*), give the binomial tri-group for five-dimensional space. Their Φ functions are respectively:

$$(1-1)^6 = 0, \quad 1-(2-1)^5 = 0, \quad (1-2)^5 + 1 = 0.$$

Cases (*v*), (*vii*), and (*viii*) each give rise to infinite figures, the first being self-reciprocal, the other two forming a reciprocal pair. The figure of Case (*v*) is infinite, for the edge of the frame figure is $\sqrt{2}$, hence that of the generating figure is equal to the radius of the circumscribed 5-fold unit sphere. In applying the criterion to Case (*vii*), we notice that the extremities of the eight edges diverging from a summit of an ikosatetrahedroid terminate at the summits of a cube, and the edges of that cube are edges of the ikosatetrahedroid. Making these eight summits coincide with those of one of the cubes of an oktahedroid inscribed to the 4-fold sphere, then the adjacent summit of the ikosatetrahedroid — since the edge of the oktahedroid is equal to the radius of the sphere — will lie at the centre of the sphere. Evidently then other ikosatetrahedroids may be added to the figure *ad infinitum*, in such a way as to saturate the four-dimensional space. The centres of these ikosatetrahedroids are the summits of an infinite series of successive hexadekahedroids, which also fill to saturation the infinite four-dimensional space. This is the figure of Case (*viii*). It may also be shown to be infinite by the application of the criterion (B).*

Cases (*iii*), (*ix*), (*vi*), (*x*), (*xi*). The edge of the frame figure of Case (*iii*) being 1.58113, the calculated length of edge of the generating figure is .97722, or less than the radius of the circumscribed sphere. Hence the figure is imaginary and also its reciprocal of Case (*ix*).† Moreover, since the edge of the frame figure of Case (*vi*) is shorter than the corresponding edge in Case (*iii*),

tical. The edge of the (120)-hedroid inscribed in the unit 4-fold sphere must be less than the edge of a regular dodekahedron inscribed in the unit 3-fold sphere; for the dodekahedral boundaries to the (120)-hedroid are themselves inscribed in *small* spheres whose radii are less than the radius of the circumscribed 4-fold sphere. Hence the limit $l < 1$ for the (120)-hedroid is justified. The limit $l < 1$ for the (600)-hedroid is verified in the last footnote to this page.

* The fact here brought to light that four-dimensional space may be built up with either hexadekahedroids or ikosatetrahedroids suggests another method for calculating the length of an edge of the ikosatetrahedroid. Let A and B (Fig. 22) be the centres of two adjacent hexadekahedroids out of 24 which have been joined together about a point O, the centre of a 4-fold sphere which circumscribes the ikosatetrahedroid whose summits are A, B, etc. The radius of the sphere is the same as of that which circumscribes the hexadekahedroid whose centre is at B, so that $OB = BF = r$. Let C be the centre of a tetrahedral boundary common to the two adjacent hexadekahedroids. Then C bisects AB and CF is perpendicular to AB. Now HF is an edge of the hexadekahedroids and is equal to $r\sqrt{2}$; and $AB = l$ is an edge of the ikosatetrahedroid; and we have

$$CF = HF\sqrt{\tfrac{3}{8}} = r\sqrt{\tfrac{3}{4}}\,;\; CF^2 + CB^2 = FB^2, \text{ or } \tfrac{3}{4}r^2 + \tfrac{1}{4}l^2 = r^2\,;\; \therefore\ l = r.$$

† This verifies the assumption on page 12, that the length of edge of the regular 4-fold (600)-hedroid is less than the radius of the circumscribed 4-fold sphere. For if it were equal to or greater than the radius of the sphere, the application of Criterion (B) to Case (*ix*) would give either an infinite or real figure.

the reciprocal figures of Cases (vi) and (x) are also imaginary; for the same reason the self-reciprocal figure of Case (xi) is imaginary.

Case (xii). The generating figure is bounded by tetrahedra, the edge of which (in applying the criterion) must be equal to the edge of the frame figure; i. e. it is less than the radius of the sphere. Hence the figure is imaginary.

Thus, according to the criterion (B) here used, there are only three regular 5-fold (simple) figures. Now if this be so, then, according to the same criterion, there can be no more than three belonging to any space of more than five dimensions. Suppose, in fact, that we have a set of the tri-group of n-fold figures, and wish to build up $(n+1)$-fold regular figures. The three figures in question, the n-fold $(n+1)$-hedroid, $(2n)$-hedroid and (2^n)-hedroid, have respectively for the numbers of the summits belonging to their $(n-1)$-fold boundaries:

$$(a)\quad (n)_0^{n-1} = n\,; \qquad (b)\quad (n)_0^{n-1} = 2^{n-1}; \qquad (c)\quad (n)_0^{n-1} = n.$$

Now the equation of Criterion (C), page 8, gives

$$(n)_1^0 N_0 = (n)_0^1 N_1 = 2\,N_1\,,$$

where $(n)_1^0$ is the number of edges radiating from a given summit, and we have for the first case $(n)_1^0(n+1) = 2\,\frac{n(n+1)}{2}$, for the second $(n)_1^0\,2^n = 2\,n\,2^{n-1}$, and for the third $(n)_1^0 \,.\, 2\,n = 2\,.\,2\,n(n-1)$, which give the group of values

$$(a)'\quad (n)_1^0 = n\,; \qquad (b)'\quad (n)_1^0 = n\,; \qquad (c)'\quad (n)_1^0 = 2\,(n-1).$$

For those arrangements in the formation of a regular angle which are possible to all spaces we are to select, in obedience to the criterion (A), the pairs of values $(n)_0^{n-1}$, $(n)_1^0$, which are identical in the two groups, and these are

$$(a)\,(a)' \qquad (a)\,(b)' \qquad (c)\,(a)' \qquad (c)\,(b)'.$$

The fourth pair gives the arrangement for the infinite self-reciprocal figure, and the other three give rise to the tri-group of regular figures in $(n+1)$-dimensional space. The other five combinations, $(a)\,(c)'$, $(b)\,(a)'$, $(b)\,(b)'$, $(b)\,(c)'$, $(c)\,(c)'$, are possible only when $n = 2, 1, 1, 2, 2$ respectively. Hence, if n be greater than 2, the only regular figures that can be produced out of combinations of the three figures of the tri-group are their own analogues in the higher spaces.

It will be obvious to the reader of this paper, that the methods herein employed are extremely liable to errors which might materially modify the conclusions drawn, and I shall be surprised if none are found;

> "For hard, hard, hard is it only not to tumble,
> So fantastical is the dainty metre."

I wish, in conclusion, to make grateful acknowledgment to my coworkers at this University, and especially to Dr. Story, for valuable suggestions.

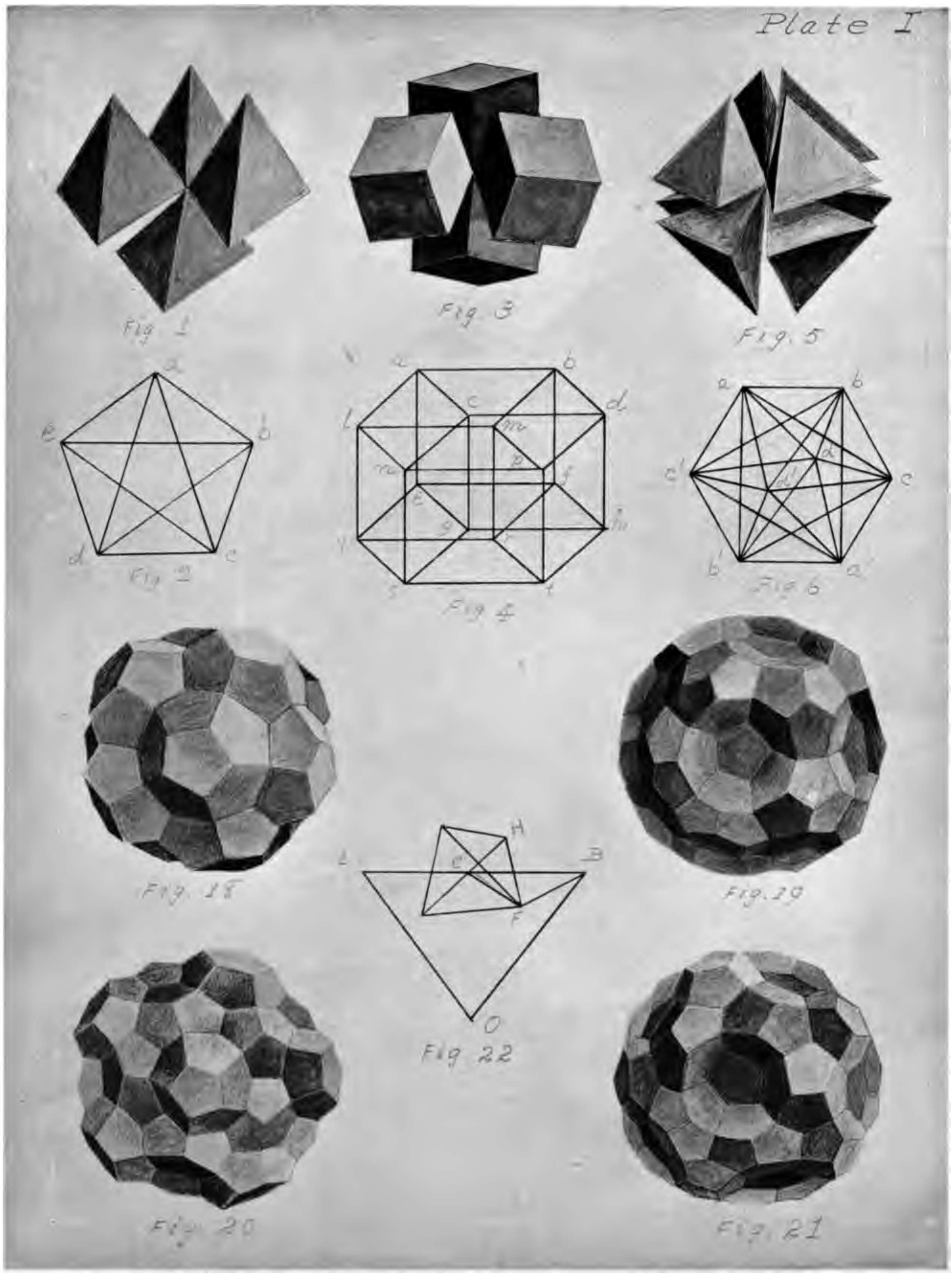
Plate I
Fig. 1
Fig. 3
Fig. 5
Fig. 2
Fig. 4
Fig. 6
Fig. 18
Fig. 19
Fig. 22
Fig. 20
Fig. 21

Plate II.

Fig. [illegible]

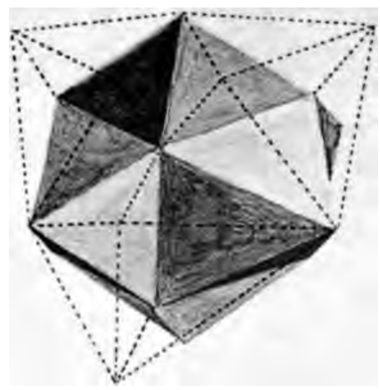

Fig. [illegible]

Fig. [illegible]

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Fig. [illegible]

Fig. [illegible]

Fig. 14

Fig. 18

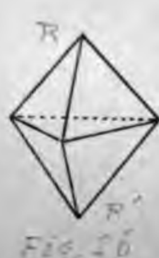


Fig. 16

Fig. 15

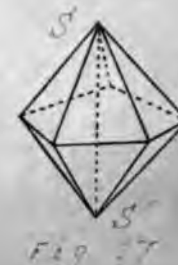


Fig. 17

On the Algebra of Logic.

By C. S. Peirce.

Chapter I. — Syllogistic.

§ 1. *Derivation of Logic.*

In order to gain a clear understanding of the origin of the various signs used in logical algebra and the reasons of the fundamental formulæ, we ought to begin by considering how logic itself arises.

Thinking, as cerebration, is no doubt subject to the general laws of nervous action.

When a group of nerves are stimulated, the ganglions with which the group is most intimately connected on the whole are thrown into an active state, which in turn usually occasions movements of the body. The stimulation continuing, the irritation spreads from ganglion to ganglion (usually increasing meantime). Soon, too, the parts first excited begin to show fatigue; and thus for a double reason the bodily activity is of a changing kind. When the stimulus is withdrawn, the excitement quickly subsides.

It results from these facts that when a nerve is affected, the reflex action, if it is not at first of the sort to remove the irritation, will change its character again and again until the irritation is removed; and then the action will cease.

Now, all vital processes tend to become easier on repetition. Along whatever path a nervous discharge has once taken place, in that path a new discharge is the more likely to take place.

Accordingly, when an irritation of the nerves is repeated, all the various actions which have taken place on previous similar occasions are the more likely to take place now, and those are most likely to take place which have most frequently taken place on those previous occasions. Now, the various actions which did not remove the irritation may have previously sometimes been performed and sometimes not; but the action which removes the irritation must

have always been performed, because the action must have every time continued until it was performed. Hence, a strong habit of responding to the given irritation in this particular way must quickly be established.

A habit so acquired may be transmitted by inheritance.

One of the most important of our habits is that one by virtue of which certain classes of stimuli throw us at first, at least, into a purely cerebral activity.

Very often it is not an outward sensation but only a fancy which starts the train of thought. In other words, the irritation instead of being peripheral is visceral. In such a case the activity has for the most part the same character; an inward action removes the inward excitation. A fancied conjuncture leads us to fancy an appropriate line of action. It is found that such events, though no external action takes place, strongly contribute to the formation of habits of really acting in the fancied way when the fancied occasion really arises.

A cerebral habit of the highest kind, which will determine what we do in fancy as well as what we do in action, is called a *belief*. The representation to ourselves that we have a specified habit of this kind is called a *judgment*. A belief-habit in its development begins by being vague, special, and meagre; it becomes more precise, general, and full, without limit. The process of this development, so far as it takes place in the imagination, is called *thought*. A judgment is formed; and under the influence of a belief-habit this gives rise to a new judgment, indicating an addition to belief. Such a process is called an *inference;* the antecedent judgment is called the *premise;* the consequent judgment, the *conclusion;* the habit of thought, which determined the passage from the one to the other (when formulated as a proposition), the *leading principle.*

At the same time that this process of inference, or the spontaneous development of belief, is continually going on within us, fresh peripheral excitations are also continually creating new belief-habits. Thus, belief is partly determined by old beliefs and partly by new experience. Is there any law about the mode of the peripheral excitations? The logician maintains that there is, namely, that they are all adapted to an end, that of carrying belief, in the long run, toward certain predestinate conclusions which are the same for all men. This is the faith of the logician. This is the matter of fact, upon which all maxims of reasoning repose. In virtue of this fact, what is to be believed at last is independent of what has been believed hitherto, and therefore has the character of *reality*. Hence, if a given habit, considered as determining an inference, is of such a sort as to tend toward the final result, it is correct; otherwise not. Thus, inferences become divisible into the valid and the invalid; and thus logic takes its reason of existence.

§ 2. *Syllogism and Dialogism.*

The general type of inference is

$$\begin{array}{c} \text{P} \\ \therefore \text{C}, \end{array}$$

where $\therefore$ is the sign of illation.

The passage from the premise (or set of premises) P to the conclusion C takes place according to a habit or rule active within us. All the inferences which that habit would determine when once the proper premises were admitted, form a class. The habit is logically good provided it would never (or in the case of a probable inference, seldom) lead from a true premise to a false conclusion; otherwise it is logically bad. That is, every possible case of the operation of a good habit would either be one in which the premise was false or one in which the conclusion would be true; whereas, if a habit of inference is bad, there is a possible case in which the premise would be true, while the conclusion was false. When we speak of a *possible* case, we conceive that from the general description of cases we have struck out all those kinds which we know how to describe in general terms but which we know never will occur; those that then remain, embracing all whose non-occurrence we are not certain of, together with all those whose non-occurrence we cannot explain on any general principle, are called possible.

A habit of inference may be formulated in a proposition which shall state that every proposition c, related in a given general way to any true proposition p, is true. Such a proposition is called the *leading principle* of the class of inferences whose validity it implies. When the inference is first drawn, the leading principle is not present to the mind, but the habit it formulates is active in such a way that, upon contemplating the believed premise, by a sort of perception the conclusion is judged to be true.* Afterwards, when the inference is subjected to logical criticism, we make a new inference, of which one premise is that leading principle of the former inference, according to which propositions related to one another in a certain way are fit to be premise and conclusion of a valid inference, while another premise is a fact of observation, namely, that the given relation does subsist between the premise and conclusion of the inference under criticism; whence it is concluded that the inference was valid.

Logic supposes inferences not only to be drawn, but also to be subjected to criticism; and therefore we not only require the form P $\therefore$ C to express an argu-

* Though the leading principle itself is not present to the mind, we are generally conscious of inferring on some general principle.

ment, but also a form, $P_1 \prec C_1$, to express the truth of its leading principle. Here P_1 denotes any one of the class of premises, and C_1 the corresponding conclusion. The symbol $\prec$ is the copula, and signifies primarily that every state of things in which a proposition of the class P_1 is true is a state of things in which the corresponding propositions of the class C_1 are true. But logic also supposes some inferences to be invalid, and must have a form for denying the leading premise. This we shall write $P_1 \overline{\prec} C_1$, *a dash over any symbol signifying in our notation the negative of that symbol.**

Thus, the form $P_1 \prec C_1$ implies

either, 1, that it is impossible that a premise of the class P_1 should be true,

or, 2, that every state of things in which P_1 is true is a state of things in which the corresponding C_1 is true.

The form $P_1 \overline{\prec} C_1$ implies

both, 1, that a premise of the class P_1 is possible,

and, 2, that among the possible cases of the truth of a P_1 there is one in which the corresponding C_1 is not true.

This acceptation of the copula differs from that of other systems of syllogistic in a manner which will be explained below in treating of the negative.

In the form of inference P ∴ C the leading principle is not expressed; and the inference might be justified on several separate principles. One of these, however, $P_1 \prec C_1$, is the formulation of the habit which, in point of fact, has governed the inferences. This principle contains all that is necessary besides the premise P to justify the conclusion. (It will generally assert more than is necessary.) We may, therefore, construct a new argument which shall have for its premises the two propositions P and $P_1 \prec C_1$ taken together, and for its conclusion, C. This argument, no doubt, has, like every other, its leading principle, because the inference is governed by some habit; but yet the substance of the leading principle must already be contained implicitly in the premises, because the proposition $P_1 \prec C_1$ contains by hypothesis all that is requisite to justify the inference of C from P. Such a leading principle, which contains no fact not implied or observable in the premises, is termed a *logical* principle, and the argument it governs is termed a *complete*, in contradistinction to an *incomplete*, argument, or *enthymeme*.

The above will be made clear by an example. Let us begin with the enthymeme,

Enoch was a man,
∴ Enoch died.

* This dash was used by Boole, but not over other than class-signs.

The leading principle of this is, "All men die." Stating it, we get the complete argument,

All men die,
Enoch was a man;
∴ Enoch was to die.

The leading principle of this is *nota notae est nota rei ipsius.* Stating this as a premise, we have the argument,

Nota notae est nota rei ipsius,
Mortality is a mark of humanity, which is a mark of Enoch;
∴ Mortality is a mark of Enoch.

But this very same principle of the *nota notae* is again active in the drawing of this last inference, so that the last state of the argument is no more complete than the last but one.

There is another way of rendering an argument complete, namely, instead of adding the leading principle $P_1 \prec C_1$ conjunctively to the premise P, to form a new argument, we might add its denial disjunctively to the conclusion; thus,

$$\begin{array}{c} P \\ \therefore \text{ Either } C \text{ or } P_1 \overline{\prec} C_1. \end{array}$$

A logical principle is said to be an *empty* or merely formal proposition, because it can add nothing to the premises of the argument it governs, although it is relevant; so that it implies no fact except such as is presupposed in all discourse, as we have seen in § 1 that certain facts are implied. We may here distinguish between *logical* and *extralogical* validity; the former being that of a *complete,* the latter that of an *incomplete* argument. The term *logical leading principle* we may take to mean the principle which must be supposed true in order to sustain the logical validity of any argument. Such a principle states that among all the states of things which can be supposed without conflict with logical principles, those in which the premise of the argument would be true would also be cases of the truth of the conclusion. Nothing more than this would be relevant to the *logical leading principle,* which is, therefore, perfectly determinate and not vague, as we have seen an extralogical leading principle to be.

A complete argument, with only one premise, is called an *immediate* inference. *Example:* All crows are black birds; therefore, all crows are birds. If from the premise of such an argument everything redundant is omitted, the state of things expressed in the premise is the same as the state of things expressed in the conclusion, and only the form of expression is changed. Now, the logician does not undertake to enumerate all the ways of expressing facts:

he supposes the facts to be already expressed in certain standard or canonical forms. But the equivalence between different ones of his own standard forms is of the highest importance to him, and thus certain immediate inferences play the great part in formal logic. Some of these will not be reciprocal inferences or logical equations, but the most important of them will have that character.

If one fact has such a relation to a different one that, if the former be true, the latter is necessarily or probably true, this relation constitutes a determinate fact; and therefore, since the leading principle of a complete argument involves no matter of fact (beyond those employed in all discourse), it follows that every complete and *material* (in opposition to a merely *formal*) argument must have at least two premises.

From the doctrine of the leading principle it appears that if we have a valid and complete argument from more than one premise, we may suppress all premises but one and still have a valid but incomplete argument. This argument is justified by the suppressed premises; hence, from these premises alone we may infer that the conclusion would follow from the remaining premises. In this way, then, the original argument

$$\begin{array}{c} \mathrm{P\ Q\ R\ S\ T} \\ \therefore \mathrm{C} \end{array}$$

is broken up into two, namely, 1st,

$$\begin{array}{c} \mathrm{P\ Q\ R\ S} \\ \therefore \mathrm{T} \prec \mathrm{C} \end{array}$$

and, 2d,

$$\begin{array}{c} \mathrm{T} \prec \mathrm{C} \\ \mathrm{T} \\ \therefore \mathrm{C}. \end{array}$$

By repeating this process, any argument may be broken up into arguments of two premises each. A complete argument having two premises is called a *syllogism*.*

An argument may also be broken up in a different way by substituting for the second constituent above, the form

$$\begin{array}{c} \mathrm{T} \prec \mathrm{C} \\ \therefore \text{Either C or not T.} \end{array}$$

In this way, any argument may be resolved into arguments, each of which has one premise and two alternative conclusions. Such an argument, when complete, may be called a *dialogism*.

* The general doctrine of this section is contained in my paper, *On the Classification of Arguments*, 1867.

§ 3. *Forms of Propositions.*

In place of the two expressions $A \prec B$ and $B \prec A$ taken together we may write $A = B$;* in place of the two expressions $A \prec B$ and $B \,\overline{\prec}\, A$ taken together we may write $A < B$ or $B > A$; and in place of the two expressions $A \,\overline{\prec}\, B$ and $B \,\overline{\prec}\, A$ taken together we may write $A \asymp B$.

De Morgan, in the remarkable memoir with which he opened his discussion of the syllogism (1846, p. 380), has pointed out that we often carry on reasoning under an implied restriction as to what we shall consider as possible, which restriction, applying to the whole of what is said, need not be expressed. The total of all that we consider possible is called the *universe* of discourse, and may be very limited. One mode of limiting our universe is by considering only what actually occurs, so that everything which does not occur is regarded as impossible.

The forms $A \prec B$, or A implies B, and $A \,\overline{\prec}\, B$, or A does not imply B, embrace both hypothetical and categorical propositions. Thus, to say that all men are mortal is the same as to say that if any man possesses any character whatever then a mortal possesses that character. To say, 'if A, then B' is obviously the same as to say that from A, B follows, logically or extralogically. By thus identifying the relation expressed by the copula with that of illation,

---

* There is a difference of opinion among logicians as to whether $\prec$ or $=$ is the simpler relation. But in my paper on the *Logic of Relatives*, I have strictly demonstrated that the preference must be given to $\prec$ in this respect. The term *simpler* has an exact meaning in logic; it means that whose logical depth is smaller; that is, if one conception implies another, but not the reverse, then the latter is said to be the simpler. Now to say that $A = B$ implies that $A \prec B$, but not conversely. *Ergo*, etc. It is to no purpose to reply that $A \prec B$ implies $A = (A$ that is $B)$; it would be equally relevant to say that $A \prec B$ implies $A = A$. Consider an analogous case. Logical sequence is a simpler conception than causal sequence, because every causal sequence is a logical sequence but not every logical sequence is a causal sequence; and it is no reply to this to say that a logical sequence between two facts implies a causal sequence between some two facts whether the same or different. The idea that $=$ is a very simple relation is probably due to the fact that the discovery of such a relation teaches us that instead of two objects we have only one, so that it simplifies our conception of the universe. On this account the existence of such a relation is an important fact to learn; in fact, it has the sum of the importances of the two facts of which it is compounded. It frequently happens that it is more convenient to treat the propositions $A \prec B$ and $B \prec A$ together in their form $A = B$; but it also frequently happens that it is more convenient to treat them separately. Even in geometry we can see that to say that two figures A and B are equal is to say that when they are properly put together A will cover B and B will cover A; and it is generally necessary to examine these facts separately. So, in comparing the numbers of two lots of objects, we set them over against one another, each to each, and observe that for every one of the lot A there is one of the lot B, and for every one of the lot B there is one of the lot A.

In logic, our great object is to analyze all the operations of reason and reduce them to their ultimate elements; and to make a calculus of reasoning is a subsidiary object. Accordingly, it is more philosophical to use the copula $\prec$, apart from all considerations of convenience. Besides, this copula is intimately related to our natural logical and metaphysical ideas; and it is one of the chief purposes of logic to show what validity those ideas have. Moreover, it will be seen further on that the more analytical copula does in point of fact give rise to the easiest method of solving problems of logic.

we identify the proposition with the inference, and the term with the proposition. This identification, by means of which all that is found true of term, proposition, or inference is at once known to be true of all three, is a most important engine of reasoning, which we have gained by beginning with a consideration of the genesis of logic.*

Of the two forms $A \mathrel{-\!\!<} B$ and $A \mathrel{\overline{-\!\!<}} B$, no doubt the former is the more primitive, in the sense that it is involved in the idea of reasoning, while the latter is only required in the criticism of reasoning. The two kinds of proposition are essentially different, and every attempt to reduce the latter to a special case of the former must fail. Boole attempts to express 'some men are not mortal,' in the form 'whatever men have a certain unknown character *v* are not mortal.' But the propositions are not identical, for the latter does not imply that some men have that character *v*; and, accordingly, from Boole's proposition we may legitimately infer that 'whatever mortals have the unknown character *v* are not men'; yet we cannot reason from 'some men are not mortal' to 'some mortals are not men.'† On the other hand, we can rise to a more general form under which $A \mathrel{-\!\!<} B$ and $A \mathrel{\overline{-\!\!<}} B$ are both included. For this purpose we write $A \mathrel{\overline{-\!\!<}} B$ in the form $\breve{A} \mathrel{-\!\!<} \bar{B}$, where $\breve{A}$ is *some*-A and $\bar{B}$ is *not*-B. This more general form is equivocal in so far as it is left undetermined whether the proposition would be true if the subject were impossible. When the subject is general this is the case, but when the subject is particular (i. e., is subject to the modification *some*) it is not. The general form supposes merely inclusion of the subject under the predicate. The short curved mark over the letter in the subject shows that some part of the term denoted by that letter is the subject, and that that is asserted to be in possible existence.

The modification of the subject by the curved mark and of the predicate by the straight mark gives the old set of propositional forms, viz.:

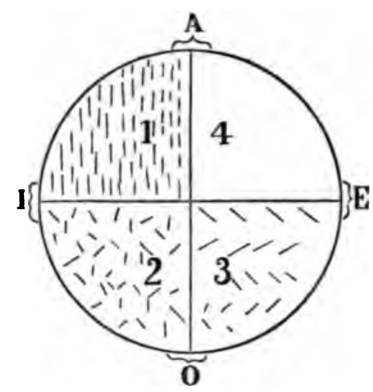
A.	$a \mathrel{-\!\!<} b$	Every a is b.	Universal affirmative.
E.	$a \mathrel{-\!\!<} \bar{b}$	No a is b.	Universal negative.
I.	$\breve{a} \mathrel{-\!\!<} b$	Some a is b.	Particular affirmative.
O.	$\breve{a} \mathrel{-\!\!<} \bar{b}$	Some a is not b.	Particular negative.

There is, however, a difference between the senses in which these propo-

* In consequence of the identification in question, in $S \mathrel{-\!\!<} P$, I speak of S indifferently as *subject*, *antecedent*, or *premise*, and of P as *predicate*, *consequent*, or *conclusion*.

† Equally unsuccessful is Mr. Jevons's attempt to overcome the difficulty by omitting particular propositions, 'because we can always substitute for it [*some*] more definite expressions if we like.' The same reason might be alleged for neglecting the consideration of *not*. But in fact the form $A \mathrel{\overline{-\!\!<}} B$ is required to enable us to simply deny $A \mathrel{-\!\!<} B$.

sitions are here taken and those which are traditional; namely, it is usually understood that affirmative propositions imply the existence of their subjects, while negative ones do not. Accordingly, it is said that there is an immediate inference from A to I and from E to O. But in the sense assumed in this paper, universal propositions do not, while particular propositions do, imply the existence of their subjects. The following figure illustrates the precise sense here assigned to the four forms A, E, I, O.



In the quadrant marked 1 there are lines which are all vertical; in the quadrant marked 2 some lines are vertical and some not; in quadrant 3 there are lines none of which are vertical; and in quadrant 4 there are no lines. Now, taking *line* as subject and *vertical* as predicate,

A is true of quadrants 1 and 4 and false of 2 and 3.
E is true of quadrants 3 and 4 and false of 1 and 2.
I is true of quadrants 1 and 2 and false of 3 and 4.
O is true of quadrants 2 and 3 and false of 1 and 4.

Hence, A and O precisely deny each other, and so do E and I. But any other pair of propositions may be either both true or both false or one true while the other is false.

De Morgan (On the Syllogism, No. I., 1846, p. 381) has enlarged the system of propositional forms by applying the sign of negation which first appears in $A \overline{\prec} B$ to the subject and predicate. He thus gets

$A \prec B$.	Every A is B.	A is species of B.
$A \overline{\prec} B$.	Some A is not B.	A is exient of B.
$A \prec \bar{B}$.	No A is B.	A is external of B.
$A \overline{\prec} \bar{B}$.	Some A is B.	A is partient of B.
$\bar{A} \prec B$.	Everything is either A or B.	A is complement of B.
$\bar{A} \overline{\prec} B$.	There is something besides A and B.	A is coinadequate of B.
$\bar{A} \prec \bar{B}$.	A includes all B.	A is genus of B.
$\bar{A} \overline{\prec} \bar{B}$.	A does not include all B.	A is deficient of B.

De Morgan's table of the relations of these propositions must be modified to conform to the meanings here attached to $\prec$ and to $\overline{\prec}$.

We might confine ourselves to the two propositional forms $S \prec P$ and $S \overline{\prec} P$. If we once go beyond this and adopt the form $S \prec \bar{P}$, we must, for

the sake of completeness, adopt the whole of De Morgan's system. But this system, as we shall see in the next section, is itself incomplete, and requires to complete it the admission of particularity in the predicate. This has already been attempted by Hamilton, with an incompetence which ought to be extraordinary. I shall allude to this matter further on, but I shall not attempt to say how many forms of propositions there would be in the completed system.*

§ 4. *The Algebra of the Copula.*

From the identity of the relation expressed by the copula with that of illation, springs an algebra. In the first place, this gives us

$$x \prec x \tag{1}$$

the principle of identity, which is thus seen to express that what we have hitherto believed we continue to believe, in the absence of any reason to the contrary. In the next place, this identification shows that the two inferences

$$\begin{array}{c} x \\ y \\ \therefore z \end{array} \qquad \text{and} \qquad \begin{array}{c} x \\ \therefore y \prec z \end{array} \tag{2}$$

are of the same validity. Hence we have

$$\{x \prec (y \prec z)\} = \{y \prec (x \prec z)\}.\dagger \tag{3}$$

From (1) we have

$$(x \prec y) \prec (x \prec y),$$

whence by (2)

$$\begin{array}{c} x \prec y \quad x \\ \therefore y \end{array} \tag{4}$$

is a valid inference.

By (4), if x and $x \prec y$ are true y is true; and if y and $y \prec z$ are true z is true. Hence, the inference is valid

$$\begin{array}{c} x \quad x \prec y \quad y \prec z \\ \therefore z. \end{array}$$

By the principle of (2) this is the same as to say that

$$\begin{array}{c} x \prec y \quad y \prec z \\ \therefore x \prec z \end{array} \tag{5}$$

is a valid inference. This is the canonical form of the syllogism, *Barbara.* The

* In this connection see De Morgan, *On the Syllogism*, No. V., 1862.

† Mr. Hugh McColl (*Calculus of Equivalent Statements*, Second Paper, 1878, p. 183) makes use of the sign of inclusion several times in the same proposition. He does not, however, give any of the formulæ of this section.

statement of its validity has been called the *dictum de omni*, the *nota notae*, etc.; but it is best regarded, after De Morgan,* as a statement that the relation signified by the copula is a transitive one.† It may also be considered as implying that in place of the subject of a proposition of the form $A \prec B$, any subject of that subject may be substituted, and that in place of its predicate any predicate of that predicate may be substituted.‡ The same principle may be algebraically conceived as a rule for the elimination of y from the two propositions $x \prec y$ and $y \prec z$.§

It is needless to remark that any letters may be substituted for x, y, z; and that, therefore, $\bar{x}$, $\bar{y}$, $\bar{z}$, some or all, may be substituted. Nevertheless, after these purely extrinsic changes have been made, the argument is no longer called *Barbara*, but is said to be some other universal mood of the *first figure*. There are evidently eight such moods.

From (5) we have, by (2), these two forms of valid immediate inference:

$$\begin{array}{c} S \prec P \\ \therefore (x \prec S) \prec (x \prec P) \end{array} \tag{6}$$

and

$$\begin{array}{c} S \prec P \\ \therefore (P \prec x) \prec (S \prec x). \end{array} \tag{7}$$

The latter may be termed the inference of *contraposition*.

From the transitiveness of the copula, the following inference is valid:

$$\begin{array}{l} (S \prec M) \prec (S \prec P) \\ (S \prec P) \prec x \\ \therefore (S \prec M) \prec x. \end{array}$$

But, by (6), from $(M \prec P)$ we can infer the first premise immediately; hence the inference is valid

$$\begin{array}{c} M \prec P \\ (S \prec P) \prec x \\ \therefore (S \prec M) \prec x. \end{array} \tag{8}$$

* *On the Syllogism*, No. II., 1850, p. 104.

† That the validity of syllogism is not deducible from the principles of identity, contradiction, and excluded middle, is capable of strict demonstration. The transitiveness of the copula is, however, implied in the identification of the copula-relation with illation, because illation is obviously transitive.

‡ The conception of substitution (already involved in the mediæval doctrine of descent), as well as the word, was familiar to logicians before the publication of Mr. Jevons's *Substitution of Similars*. This book argues, however, not only that inference is substitution, but that it and induction in particular consist in the substitution of similars. This doctrine is allied to Mill's theory of induction.

§ This must have been in Boole's mind from the first. De Morgan (*On the Syllogism*, No. II., 1850, p. 83) goes too far in saying that "what is called elimination in algebra is called inference in logic," if he means, as he seems to do, that all inference is elimination.

This may be called the *minor indirect syllogism.* The following is an example:

All men are mortal,
If Enoch and Elijah were mortal, the Bible errs;
∴ If Enoch and Elijah were men, the Bible errs.

Again we may start with this syllogism in *Barbara*

$$\begin{aligned}&(M \prec P) \prec (S \prec P),\\ &(S \prec P) \prec x;\\ \therefore\ &(M \prec P) \prec x.\end{aligned}$$

But by the principle of contraposition (7), the first premise immediately follows from $(S \prec M)$, so that we have the inference valid

$$\begin{aligned}&S \prec M,\\ &(S \prec P) \prec x;\\ \therefore\ &(M \prec P) \prec x.\end{aligned} \qquad (9)$$

This may be called the *major indirect syllogism.*

Example: All patriarchs are men,
If all patriarchs are mortal, the Bible errs;
∴ If all men are mortal, the Bible errs.

In the same way it might be shown that (6) justifies the syllogism

$$\begin{aligned}&M \prec P,\\ &x \prec (S \prec M);\\ \therefore\ &x \prec (S \prec P).\end{aligned} \qquad (10)$$

And (7) justifies the inference

$$\begin{aligned}&S \prec M,\\ &x \prec (M \prec P);\\ \therefore\ &x \prec (S \prec P).\end{aligned} \qquad (11)$$

But these are only slight modifications of *Barbara.*

In the form (10), x may denote a limited universe comprehending some cases of S. Then we have the syllogism

$$\begin{aligned}&M \prec P,\\ &S \mathrel{\overline{\prec}} \overline{M};\\ \therefore\ &S \mathrel{\overline{\prec}} \overline{P}.\end{aligned} \qquad (12)$$

This is called *Darii.* A line might, of course, be drawn over the S. So, in the form (11), x may denote a limited universe comprehending some $\overline{M}$. Then we have the syllogism

$$
\begin{aligned}
&\mathrm{S} \prec \mathrm{M}, \\
&\bar{\mathrm{M}} \mathrel{\overline{\prec}} \mathrm{P}; \\
\therefore\ &\bar{\mathrm{S}} \mathrel{\overline{\prec}} \mathrm{P}.
\end{aligned} \tag{13}
$$

Here a line might be drawn over the P. But the forms (12) and (13) are deduced from (10) and (11) only by principles of interpretation which require demonstration.

On the other hand, if in the *minor indirect syllogism* (8), we put "what does not occur" for x, we have by definition

$$\{(\mathrm{S} \prec \mathrm{P}) \prec x\} = (\mathrm{S} \mathrel{\overline{\prec}} \mathrm{P})$$

and we then have

$$
\begin{aligned}
&\mathrm{M} \prec \mathrm{P}, \\
&\mathrm{S} \mathrel{\overline{\prec}} \mathrm{P}; \\
\therefore\ &\mathrm{S} \mathrel{\overline{\prec}} \mathrm{M},
\end{aligned} \tag{14}
$$

which is the syllogism *Baroko*. If a line is drawn over P, the syllogism is called *Festino;* and by other negations eight essentially identical forms are obtained, which are called minor-particular moods of the second figure.* In the same way the major indirect syllogism (9) affords the form

$$
\begin{aligned}
&\mathrm{S} \prec \mathrm{M}, \\
&\mathrm{S} \mathrel{\overline{\prec}} \mathrm{P}; \\
\therefore\ &\mathrm{M} \mathrel{\overline{\prec}} \mathrm{P}.
\end{aligned} \tag{15}
$$

This form is called *Bocardo*. If P is negatived, it is called *Disamis*. Other negations give the eight major-particular moods of the third figure.

We have seen that $\mathrm{S} \mathrel{\overline{\prec}} \mathrm{P}$ is of the form $(\mathrm{S} \prec \mathrm{P}) \prec x$. Put A for $\mathrm{S} \prec \mathrm{P}$, and we find that $\bar{\mathrm{A}}$ is of the form $\mathrm{A} \prec x$. Then the principle of contraposition (7) gives the immediate inference

$$
\begin{aligned}
&\mathrm{S} \prec \mathrm{P} \\
\therefore\ &\bar{\mathrm{P}} \prec \bar{\mathrm{S}}.
\end{aligned} \tag{16}
$$

Applying this to the universal moods of the first figure justifies six moods. These are two in the second figure,

$$
\begin{array}{lll}
x \prec \bar{y} & z \prec y & \therefore\ x \prec \bar{z} \ (\textit{Camestres}) \\
\bar{x} \prec \bar{y} & z \prec y & \therefore\ \bar{x} \prec \bar{z};
\end{array}
$$

two in the third figure,

$$
\begin{array}{lll}
y \prec x & \bar{y} \prec z & \therefore\ \bar{x} \prec z \\
y \prec x & \bar{y} \prec \bar{z} & \therefore\ \bar{x} \prec \bar{z};
\end{array}
$$

* De Morgan, *Syllabus*, 1860, p. 18.

and two others which are said to be in the fourth figure.

$$x \prec y \qquad y \prec z \qquad \therefore\ \bar{z} \prec \bar{x}$$
$$x \prec \bar{y} \qquad \bar{y} \prec z \qquad \therefore\ \bar{z} \prec \bar{x}.$$

But the negative has two other properties not yet taken into account. These are

$$x \prec \bar{\bar{x}} \tag{17}$$

or x is not not-X, which is called the *principle of contradiction*: and

$$\bar{\bar{x}} \prec x \tag{18}$$

or what is not not-X is x, which is called the *principle of excluded middle*.

By (17) and (16) we have the immediate inference

$$\begin{array}{l} \mathrm{S} \prec \bar{\mathrm{P}} \\ \therefore\ \mathrm{P} \prec \bar{\mathrm{S}} \end{array} \tag{19}$$

which is called the conversion of E. By (18) and (16) we have

$$\begin{array}{l} \bar{\mathrm{S}} \prec \mathrm{P} \\ \therefore\ \bar{\mathrm{P}} \prec \mathrm{S}. \end{array} \tag{20}$$

By (17), (18), and (16), we have

$$\begin{array}{l} \bar{\mathrm{S}} \prec \bar{\mathrm{P}} \\ \therefore\ \mathrm{P} \prec \mathrm{S}. \end{array} \tag{21}$$

Each of the inferences (19), (20), (21), justifies six universal syllogisms; namely, two in each of the figures, second, third, and fourth. The result is that each of these figures has eight universal moods; two depending only on the principle that $\bar{\mathrm{A}}$ is of the form $\mathrm{A} \prec x$, two depending also on the principle of contradiction, two on the principle of excluded middle, and two on all three principles conjoined.

The same formulæ (16), (19), (20), (21), applied to the minor-particular moods of the second figure, will give eight minor-particular moods of the first figure; and applied to the major-particular moods of the third figure, will give eight major-particular moods of the first figure.*

The principle of contradiction in the form (19) may be further transformed thus: —

$$\text{If } (\mathrm{P} \therefore \bar{\mathrm{C}}) \text{ is valid, then } (\mathrm{C} \therefore \bar{\mathrm{P}}) \text{ is valid.} \tag{22}$$

Applying this to the minor-particular moods of the first figure, will give eight minor-particular moods of the third figure; and applying it to the major-particu-

* Aristotle and De Morgan have particular conclusions from two universal premises. These are all rendered illogical by the significations which I attach to $\prec$ and $\overline{\prec}$.

lar moods of the first figure will give eight major-particular moods of the second figure.

It is very noticeable that the corresponding formula,

$$\text{If } (\bar{\mathrm{P}} \mathrel{.\cdot.} \mathrm{C}) \text{ is valid, then } (\bar{\mathrm{C}} \mathrel{.\cdot.} \mathrm{P}) \text{ is valid,} \tag{23}$$

has no application in the existing syllogistic, because there are no syllogisms having a particular premise and universal conclusion. In the same way, in the Aristotelian system an affirmative conclusion cannot be drawn from negative premises, the reason being that negation is only applied to the predicate. So in De Morgan's system the subject only is made particular, not the predicate.

In order to develop a system of propositions in which the predicate shall be modified in the same way in which the subject is modified in particular propositions, we should consider that to say $\mathrm{S} \prec \mathrm{P}$ is the same as to say $(\mathrm{S} \mathrel{\overline{\prec}} x) \prec (\mathrm{P} \mathrel{\overline{\prec}} x)$, whatever x may be. That

$$(\mathrm{S} \prec \mathrm{P}) \prec \{(\mathrm{S} \mathrel{\overline{\prec}} x) \prec (\mathrm{P} \mathrel{\overline{\prec}} x)\}$$

follows at once from *Bokardo* (15) by means of (2). Moreover, since $\bar{\mathrm{A}}$ may be put in the form $\mathrm{A} \prec x$, it follows that $\bar{\bar{\mathrm{A}}}$ may be put in the form $\mathrm{A} \mathrel{\overline{\prec}} x$, so that by the principles of contradiction and excluded middle, A may be put in the form $\mathrm{A} \mathrel{\overline{\prec}} x$. On the other hand, to say $\mathrm{S} \mathrel{\overline{\prec}} \bar{\mathrm{P}}$ is the same as to say $(\mathrm{S} \prec \bar{x}) \prec (\mathrm{P} \mathrel{\overline{\prec}} x)$, whatever x may be; for

$$(\mathrm{S} \mathrel{\overline{\prec}} \bar{\mathrm{P}}) \prec \{(\mathrm{S} \prec \bar{x}) \prec (\mathrm{P} \mathrel{\overline{\prec}} x)\}$$

is the principle of *Ferison*, a valid syllogism of the third figure; and if for x we put $\bar{\mathrm{S}}$, we have

$$(\mathrm{S} \prec \bar{\bar{\mathrm{S}}}) \prec (\mathrm{P} \mathrel{\overline{\prec}} \bar{\mathrm{S}}),$$

which is the same as to say that $\mathrm{P} \mathrel{\overline{\prec}} \bar{\mathrm{S}}$ is true if the principle of contradiction is true. So that it follows that $\mathrm{P} \mathrel{\overline{\prec}} \bar{\mathrm{S}}$ if $\mathrm{S} \mathrel{\overline{\prec}} \bar{\mathrm{P}}$ from the principle of contradiction. Comparing

$$\mathrm{S} \prec \mathrm{P} \quad \text{or} \quad (\mathrm{S} \mathrel{\overline{\prec}} x) \prec (\mathrm{P} \mathrel{\overline{\prec}} x)$$

with

$$\mathrm{S} \mathrel{\overline{\prec}} \bar{\mathrm{P}} \quad \text{or} \quad (\mathrm{S} \prec \bar{x}) \prec (\mathrm{P} \mathrel{\overline{\prec}} x),$$

we see that they differ by a modification of the subject. Denoting this by a short curve over the subject, we may write $\breve{\mathrm{S}} \prec \mathrm{P}$ for $\mathrm{S} \mathrel{\overline{\prec}} \bar{\mathrm{P}}$. We see then that while for A we may write $\mathrm{A} \mathrel{\overline{\prec}} x$, where x is anything whatever, so for $\breve{\mathrm{A}}$ we may write $\mathrm{A} \prec \bar{x}$. If we attach a similar modification to the predicate also, we have

$$\breve{\mathrm{S}} \prec \breve{\mathrm{P}} \quad \text{or} \quad (\mathrm{S} \prec \bar{x}) \prec (\mathrm{P} \prec \bar{x}),$$

which is the same as to say that you can find an S which is any P you please. We thus have

$$(\mathrm{S} -\!\!< \mathrm{P}) -\!\!< (\breve{\mathrm{P}} -\!\!< \breve{\mathrm{S}}), \qquad (24)$$

a formula of contraposition, similar to (16).

It is obvious that

$$(\breve{\mathrm{S}} -\!\!< \mathrm{P}) -\!\!< (\breve{\mathrm{P}} -\!\!< \mathrm{S})\,; \qquad (25)$$

for, negating both propositions, this becomes, by (16),

$$(\mathrm{P} -\!\!< \bar{\mathrm{S}}) -\!\!< (\mathrm{S} -\!\!< \bar{\mathrm{P}}),$$

which is (19). The inference justified by (25) is called the conversion of I. From (25) we infer

$$\breve{\breve{x}} -\!\!< x, \qquad (26)$$

which may be called the principle of particularity. This is obviously true, because the modification of particularity only consists in changing $(\mathrm{A} \;\overline{-\!\!<}\; x)$ to $(\mathrm{A} -\!\!< \bar{x})$, which is the same as negating the copula and predicate, and a repetition of this will evidently give the first expression again. For the same reason we have

$$x -\!\!< \breve{\breve{x}}, \qquad (27)$$

which may be called the principle of individuality. This gives

$$(\mathrm{S} -\!\!< \breve{\mathrm{P}}) -\!\!< (\mathrm{P} -\!\!< \breve{\mathrm{S}}), \qquad (28)$$

and (26) and (27) together give

$$(\breve{\mathrm{S}} -\!\!< \breve{\mathrm{P}}) -\!\!< (\mathrm{P} -\!\!< \mathrm{S}). \qquad (29)$$

It is doubtful whether the proposition $\mathrm{S} -\!\!< \breve{\mathrm{P}}$ ought to be interpreted as signifying that S and P are one sole individual, or that there is something besides S and P. I here leave this branch of the subject in an unfinished state.

Corresponding to the formulæ which we have obtained by the principle (2) are an equal number obtained by the following principle:

(2′) The inference

$$\begin{array}{c} x \\ \therefore\ \text{Either } y \text{ or } z \end{array}$$

has the same validity as

$$\begin{array}{c} x \;\overline{-\!\!<}\; y \\ \therefore\ z. \end{array}$$

From (1) we have

$$(x \;\overline{-\!\!<}\; y) -\!\!< (x \;\overline{-\!\!<}\; y),$$

whence, by (2),

(4′)
$$\begin{gathered} x \\ \therefore \text{ Either } (x \prec y) \text{ or } y. \end{gathered}$$

This gives

$$\begin{gathered} x \\ \therefore \text{ Either } x \overline{\prec} y \text{ or } y \overline{\prec} z \text{ or } z. \end{gathered}$$

Then, by (2),

(5′)
$$\begin{gathered} x \overline{\prec} z \\ \therefore\ x \overline{\prec} y \text{ or } y \overline{\prec} z, \end{gathered}$$

which is the canonical form of dialogism. The minor indirect dialogism is

(8′)
$$\begin{gathered} x \overline{\prec} (\mathrm{M} \overline{\prec} \mathrm{P}) \\ \therefore \text{ Either } x \overline{\prec} (\mathrm{S} \overline{\prec} \mathrm{P}) \text{ or } \mathrm{S} \overline{\prec} \mathrm{M}. \end{gathered}$$

The major indirect dialogism is

$$\begin{gathered} x \overline{\prec} (\mathrm{S} \overline{\prec} \mathrm{M}) \\ \therefore \text{ Either } x \overline{\prec} (\mathrm{S} \overline{\prec} \mathrm{P}) \text{ or } \mathrm{M} \overline{\prec} \mathrm{P}. \end{gathered}$$

We have also

(12′)
$$\begin{gathered} (\mathrm{S} \overline{\prec} \mathrm{P}) \overline{\prec} x \\ \therefore \text{ Either } (\mathrm{S} \overline{\prec} \mathrm{M}) \text{ or } (\mathrm{M} \overline{\prec} \mathrm{P}) \overline{\prec} x \end{gathered}$$

and

(13′)
$$\begin{gathered} (\mathrm{S} \overline{\prec} \mathrm{P}) \overline{\prec} x \\ \therefore \text{ Either } (\mathrm{M} \overline{\prec} \mathrm{P}) \text{ or } (\mathrm{S} \overline{\prec} \mathrm{M}) \overline{\prec} x. \end{gathered}$$

We have A of the form $x \overline{\prec} \bar{\mathrm{A}}$. And we have the inferences

$$\begin{array}{cccc} \mathrm{S} \overline{\prec} \mathrm{P} & \mathrm{S} \overline{\prec} \bar{\mathrm{P}} & \bar{\mathrm{S}} \overline{\prec} \mathrm{P} & \bar{\mathrm{S}} \overline{\prec} \bar{\mathrm{P}} \\ \therefore\ \bar{\mathrm{P}} \overline{\prec} \bar{\mathrm{S}}. & \therefore\ \mathrm{P} \overline{\prec} \bar{\mathrm{S}}. & \therefore\ \bar{\mathrm{P}} \overline{\prec} \mathrm{S}. & \therefore\ \mathrm{P} \overline{\prec} \mathrm{S}. \end{array}$$

CHAPTER II.—THE LOGIC OF NON-RELATIVE TERMS.

§ 1. *The Internal Multiplication and the Addition of Logic.*

We have seen that the inference

$$\begin{gathered} x \text{ and } y \\ \therefore\ z \end{gathered}$$

is of the same validity with the inference

$$\begin{gathered} x \\ \therefore \text{ Either } \bar{y} \text{ or } z, \end{gathered}$$

and the inference

$$x$$
$$\therefore \text{ Either } y \text{ or } z$$

with the inference

$$x \text{ and } \bar{y}$$
$$\therefore z.$$

In like manner,

$$x \prec y$$

is equivalent to

$$(\text{The possible}) \prec \text{Either } \bar{x} \text{ or } y,$$

and to

$$x \text{ which is } \bar{y} \prec (\text{The impossible}).$$

To express this algebraically, we need, in the first place, symbols for the two terms of second intention, the possible and the impossible. Let ∞ and 0 be the terms; then we have the definitions

$$x \prec \infty \qquad 0 \prec x \tag{1}$$

whatever x may be.*

We need also two operations which may be called non-relative addition and multiplication. They are defined as follows: †

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* The symbol 0 is used by Boole; the symbol ∞ replaces his 1, according to a suggestion in my *Logic of Relatives*, 1870.

† These forms of definition are original. The algebra of non-relative terms was given by Boole (*Mathematical Analysis of Logic*, 1847). Boole's addition was not the same as that in the text, for with him whatever was common to the two terms added was taken twice over in the sum. The operations in the text were given as complements of one another, and with appropriate symbols, by De Morgan (*On the Syllogism*, No. III., 1858, p. 185). For addition, sum, parts, he uses aggregation, aggregate, aggregants; for multiplication, product, factors, he uses composition, compound, components. Mr. Jevons (*Formal Logic*, 1864) — I regret that I can only speak of this work from having read it many years ago, and therefore cannot be sure of doing it full justice — improved the algebra of Boole by substituting De Morgan's aggregation for Boole's addition. The present writer, not having seen either De Morgan's or Jevons's writings on the subject, again recommended the same change (*On an Improvement in Boole's Calculus of Logic*, 1867), and showed the perfect balance existing between the two operations. In another paper, published in 1870, I introduced the sign of inclusion into the algebra.

In 1872, Robert Grassmann, brother of the author of the *Ausdehnungslehre*, published a work entitled '*Die Formenlehre oder Mathematik*,' the second book of which gives an algebra of logic identical with that of Jevons. The very notation is reproduced, except that the universe is denoted by T instead of U, and a term is negatived by drawing a line over it, as by Boole, instead of by taking a type from the other case, as Jevons does. Grassmann also uses a sign equivalent to my $\prec$. In his third book, he has other matter which he might have derived from my paper of 1870. Grassmann's treatment of the subject presents inequalities of strength; and most of his results had been anticipated. Professor Schröder, of Karlsruhe, in the spring of 1877, produced his *Operationskreis des Logikkalkuls*. He had seen the works of Boole and Grassmann, but not those of De Morgan, Jevons, and me. He gives a fine development of the algebra, adopting the addition of Jevons, and he exhibits the balance between $+$ and $\times$ by printing the theorems in parallel columns, thus imitating a practice of the geometricians. Schröder gives an original, interesting, and commodious method of working with the algebra. Later in the same year, Mr. Hugh McColl, apparently having known nothing of logical algebra except from a jejune account of Boole's work in *Bain's Logic*, published several papers on a *Calculus of Equivalent Statements*, the basis of which is nothing but the Boolian algebra, with Jevons's addition and a sign of inclusion. Mr. McColl adds an exceedingly ingenious application of this algebra to the transformation of definite integrals.

If $a \prec x$ and $b \prec x$, then $a + b \prec x$;	If $x \prec a$ and $x \prec b$, then $x \prec a \times b$; (2)
and conversely,	and conversely,
if $a + b \prec x$, then $a \prec x$ and $b \prec x$.	if $x \prec a \times b$, then $x \prec a$ and $x \prec b$. (3)

From these definitions we at once deduce the following formulæ: —

A. $\quad a \prec a + b \qquad a \times b \prec a$ (Peirce, 1870) *
$\quad b \prec a + b \qquad a \times b \prec b$. (4)

These are proved by substituting $a + b$ and $a \times b$ for x in (3).

B. $\quad x = x + x \qquad x \times x = x$ (Jevons, 1864). (5)

By substituting x for a and b in (2), we get

$$x + x \prec x \qquad x \prec x \times x;$$

and, by (4),

$$x \prec x + x \qquad x \times x \prec x.$$

C. $\quad a + b = b + a \qquad a \times b = b \times a$ (Boole, Jevons). (6)

These formulæ are examples of the *commutative principle*. From (4) and (2),

$$b + a \prec a + b \qquad a \times b \prec b \times a$$

and interchanging a and b we get the reciprocal inclusion implied in (6).

D. $\quad (a+b)+c = a+(b+c) \qquad a \times (b \times c) = (a \times b) \times c$ (Boole, Jevons). (7)

These are cases of the *associative principle*. By (4), $c \prec b + c$ and $b \times c \prec c$; also $b + c \prec a + (b + c)$ and $a \times (b \times c) \prec b \times c$; so that $c \prec a + (b + c)$ and $a \times (b \times c) \prec c$. In the same way, $b \prec a + (b + c)$ and $a \times (b \times c) \prec b$, and, by (4), $a \prec a + (b + c)$ and $a \times (b \times c) \prec a$. Hence, by (2), $a + b \prec a + (b + c)$ and $a \times (b \times c) \prec a \times b$. And, again by (2), $(a + b) + c \prec a + (b + c)$ and $a \times (b \times c) \prec (a \times b) \times c$. In a similar way we should prove the converse propositions to these and so establish (7).

E. $\quad (a+b) \times c = (a \times c) + (b \times c) \qquad (a \times b) + c = (a+c) \times (b+c)$.† (8)

These are cases of the *distributive principle*. They are easily proved by (4) and (2), but the proof is too tedious to give.

F. $\quad (a+b)+c = (a+c)+(b+c) \qquad (a \times b) \times c = (a \times c) \times (b \times c)$. (9)

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* *Logic of Relatives* (§ 4); gives $a \times b \prec a$. The other formulæ, equally obvious, I do not find anywhere.

† The first of these given by Boole for his addition, was retained by Jevons in changing the addition. The second was first given by me (1867).

These are other cases of the distributive principle. They are proved by (5), (6) and (7). These formulæ, which have hitherto escaped notice, are not without interest.

G. $a + (a \times b) = a \qquad a \times (a + b) = a$ (Grassmann, Schröder). (10)

By (4), $\quad a \prec a + (a \times b) \qquad a \times (a + b) \prec a.$

Again, by (4), $(a \times b) \prec a$ and $a \prec a + b$; hence, by (2)

$$a + (a \times b) \prec a \qquad a \prec a \times (a + b).$$

H. $\quad (a + b \prec a) = (b \prec a \times b).$ (11)

This proposition is a transformation of Schröder's two propositions 21, (p. 25), one of which was given by Grassmann. By (3)

$$(a + b \prec a) \prec (b \prec a) \qquad (b \prec a \times b) \prec (b \prec a).$$

Hence, since $\quad b \prec b, \qquad a \prec a$

we have, by (2),

$$(a + b \prec a) \prec (b \prec a \times b) \qquad (b \prec a \times b) \prec (a + b \prec a).$$

I. $\left.\begin{matrix}(a \prec b) \times (x \prec y) \prec (a + x \prec b + y) \\ (a \prec b) \times (x \prec y) \prec (a \times x \prec b \times y)\end{matrix}\right\}$ (Peirce, 1870). (12)

Readily proved from (2) and (4).

J. $\quad (a \prec b + x) \times (a \times x \prec b) = (a \prec b).$ (13)

This is a generalization of a theorem by Grassmann. In stating it, he erroneously unites the first two propositions by $+$ instead of $\times$. By (12), (5), and (8),

$$(a \prec b + x) \prec \{a \prec (a \times b) + (a \times x)\}$$
$$(a \times x \prec b) \prec \{(a + b) \times (x + b) \prec b\}.$$

But by (4)

$$a \prec a + b \qquad a \times b \prec b.$$

Hence, by (2), it is doubly proved that

$$(a \prec b + x) \times (a \times x \prec b) \prec (a \prec b).$$

The demonstration of the converse is obvious.

We have immediately, from (2) and (3),

K. $(a + b \prec c) = (a \prec c) \times (b \prec c) \quad (c \prec a \times b) = (c \prec a) \times (c \prec b)$ (14)

L. $\begin{matrix}(c \prec a + b) = \Sigma\,\{(p \prec a) \times (q \prec b)\} & \text{where } p + q = c \\ (a \times b \prec c) = \Sigma\,\{(a \prec p) \times (b \prec q)\} & \text{where } c = p \times q.\end{matrix}$ (15)

The propositions (15) are new. By (12)

$$\{(p \prec a) \times (q \prec b)\} \prec (c \prec a + b) \quad \text{where } p + q = c$$
$$\{(a \prec p) \times (b \prec q)\} \prec (a \times b \prec c) \quad \text{where } c = p \times q.$$

And, since these are true for any set of values of p and q, we have by (2)

$$\Sigma\{(p \prec a)\times(q \prec b)\} \prec (c \prec a+b), \text{ where } p+q=c.$$
$$\Sigma\{(a \prec p)\times(b \prec q)\} \prec (a\times b \prec c), \text{ where } c=p\times q.$$

By (4) and (8), we have

$$(c \prec a+b) \prec \{(a\times c)+(b\times c)=c\}$$
$$(a\times b \prec c) \prec \{(c+a)\times(c+b)=c\}.$$

Hence, putting

$$a\times c=p \qquad b\times c=q, \qquad \text{where } p+q=c$$
$$a+c=p \qquad b+c=q, \qquad \text{where } p\times q=c,$$

we have

$$(c \prec a+b) \prec (p \prec a)\times(q \prec b), \text{ where } p+q=c$$
$$(a\times b \prec c) \prec (a \prec p)\times(b \prec q), \text{ where } c=p\times q,$$

whence, by (4)

$$(c \prec a+b) \prec \Sigma\{(p \prec a)\times(q \prec b)\} \text{ where } p+q=c$$
$$(a\times b \prec c) \prec \Sigma\{(a \prec p)\times(b \prec q)\} \text{ where } c=p\times q.$$

A formula analogous to (15) will be found below, (35).

From (1) and (2) and (4) we have

$$x+0=x \qquad x=x\times\infty. \tag{16}$$

From (1) and (4),

$$x+\infty=\infty \qquad 0=x\times 0. \tag{17}$$

The definition of the negative has as we have seen three clauses: first, that $\bar{a}$ is of the form $a \prec x$; second, $a \prec \bar{\bar{a}}$; third, $\bar{\bar{a}} \prec a$.

From the first we have that if

$$\begin{array}{c} c \quad a \\ \therefore\ b \end{array}$$

is valid, then

$$\begin{array}{c} c \quad \bar{b} \\ \therefore\ \bar{a} \end{array}$$

is valid. Or

$$(c\times a \prec b) \prec (c\times\bar{b} \prec \bar{a}). \tag{18}$$

Also, that if

$$\begin{array}{c} b \\ \therefore \text{ Either } c \text{ or } a \end{array}$$

is valid, then

$$\begin{array}{c} \bar{a} \\ \therefore \text{ Either } c \text{ or } \bar{b} \end{array}$$

is valid; or

$$(b \prec c+a) \prec (\bar{a} \prec c+\bar{b}). \tag{19}$$

Combining (18) and (19), we have

$$(a\times b \prec c+d) \prec (a\times\bar{d} \prec c+\bar{b}). \tag{20}$$

By the principles of contradiction and excluded middle, this gives

$$(a \times \bar{d} \prec c + \bar{b}) \prec (a \times b \prec c + d). \tag{21}$$

Thus the formula

$$(a \times b \prec c + d) = (a \times \bar{d} \prec c + \bar{b}) \tag{22}$$

embodies the essence of the negative.

If in (22) we put, first, $a = d \quad b = c = 0$, and then $a = d = \infty \quad b = c$, we have from the formula of identity

$$a \times \bar{a} = 0 \qquad a + \bar{a} = \infty. \tag{23}$$

We have

$$p = (p \times x) + (p \times \bar{x}) \qquad p = (p + x) \times (p + \bar{x}) \tag{24}$$

by the distributive principle and (23). If we write

$$i = p + (a \times \bar{x}) \qquad j = p + (b \times x) \qquad k = p \times (c + x) \qquad l = p \times (d + \bar{x}),$$

we equally have

$$p = (i \times x) + (j \times \bar{x}) \qquad p = (l + x) \times (k + \bar{x}). \tag{25}$$

Now p may be a function of x, and such values may perhaps be assigned to a, b, c, d, that i, j, k, l, shall be free from x. It is obvious that if the function results from any complication of the operations $+$ and $\times$, this is the case. Supposing, then, i, j, k, l, to be constant, we have, putting successively, 1, and 0, for x.

$$\phi\infty = i = k$$
$$\phi 0 = j = l$$

so that

$$\phi x = (\phi\infty \times x) + (\phi 0 \times \bar{x}) \qquad \phi x = (\phi 0 + x) \times (\phi\infty + \bar{x}). \tag{26}$$

The first of these formulæ was given by Boole for his addition. I showed (1867) that both hold for the modified addition. These formulæ are real analogues of mathematical developments; but practically they are not convenient. Their connection suggests the general formula

$$(a + x) \times (b + \bar{x}) = (a \times \bar{x}) + (b \times x) \tag{27}$$

a formula of frequent utility.

The distributive principle and (3) applied to (26) give

$$\phi 0 \times \phi\infty \prec \phi x \qquad \phi x \prec \phi\infty + \phi 0. \tag{28}$$

Hence

$$(\phi x = 0) \prec (\phi 0 \times \phi\infty = 0) \qquad (\phi x = \infty) \prec (\phi 0 + \phi\infty = \infty). \tag{29}$$

Boole gave the former, and I (1867) the latter. These formulæ are not convenient for elimination.

The following formulæ (probably given by De Morgan) are of great importance: —

$$\overline{a \times b} = \bar{a} + \bar{b} \qquad \overline{a + b} = \bar{a} \times \bar{b}. \tag{30}$$

By (23)

$$(a \times b) \times (\overline{a \times b}) \prec 0 \qquad \infty \prec (a + b) + (\overline{a + b}),$$

whence by (22) and the associative principle

$$b \times (\overline{a \times b}) \prec \bar{a} \qquad \bar{a} \prec b + (\overline{a + b})$$
$$\overline{a \times b} \prec \bar{a} + \bar{b} \qquad \bar{a} \times \bar{b} \prec \overline{a + b}.$$

By (4) and (22)

$$\bar{a} \prec \overline{a \times b} \qquad \overline{a + b} \prec \bar{a}$$
$$\bar{b} \prec \overline{a \times b} \qquad \overline{a + b} \prec \bar{b},$$

whence by (2)

$$\bar{a} + \bar{b} \prec \overline{a + b} \qquad \overline{a + b} \prec \bar{a} \times \bar{b}.$$

The application of (22) gives from (11)

$$(b \mathrel{\overline{\prec}} a \times b) = (a + b \mathrel{\overline{\prec}} a); \tag{31}$$

from (12)

$$\left.\begin{aligned} (a + x \mathrel{\overline{\prec}} b + y) &\prec (a \mathrel{\overline{\prec}} b) + (x \mathrel{\overline{\prec}} y) \\ (a \times x \mathrel{\overline{\prec}} b \times y) &\prec (a \mathrel{\overline{\prec}} b) + (x \mathrel{\overline{\prec}} y); \end{aligned}\right\} \tag{32}$$

from (13)

$$(a \mathrel{\overline{\prec}} b) = (a \mathrel{\overline{\prec}} b + x) + (a \times x \mathrel{\overline{\prec}} b); \tag{33}$$

from (14)

$$(a + b \mathrel{\overline{\prec}} c) = (a \mathrel{\overline{\prec}} c) + (b \mathrel{\overline{\prec}} c) \qquad (c \mathrel{\overline{\prec}} a \times b) \prec (c \mathrel{\overline{\prec}} a) + (c \mathrel{\overline{\prec}} b); \tag{34}$$

from (15)

$$\left.\begin{aligned} (c \mathrel{\overline{\prec}} a + b) &= \Pi \{ (p \mathrel{\overline{\prec}} a) + (q \mathrel{\overline{\prec}} b) \} \text{ where } p + q = c \\ (a \times b \mathrel{\overline{\prec}} c) &= \Pi \{ (a \mathrel{\overline{\prec}} p) + (b \mathrel{\overline{\prec}} q) \} \text{ where } p \times q = c; \end{aligned}\right\} \tag{35}$$

from (22)

$$(a \times b \mathrel{\overline{\prec}} c + d) = (a \times \bar{d} \mathrel{\overline{\prec}} c + \bar{b}). \tag{36}$$

§ 2. *The Resolution of Problems in Non-relative Logic.*

Four different algebraic methods of solving problems in the logic of non-relative terms have already been proposed by Boole, Jevons, Schröder, and McColl. I propose here a fifth method which perhaps is simpler and certainly is more natural than any of the others. It involves the following processes:

First Process. Express all the premises with the copulas $\prec$ and $\overline{\prec}$, remembering that $A = B$ is the same as $A \prec B$ and $B \prec A$.

Second Process. Separate every predicate into as many factors and every subject into as many aggregant terms as is possible without increasing the number of different letters used in any subject or predicate.

An expression might be separated into such factors or aggregants (let us term them *prime* factors and *ultimate* aggregants) by one or other of these formulæ:

$$\phi x = (\phi \infty \times x) + (\phi 0 \times \bar{x})$$
$$\phi x = (\phi \infty + \bar{x}) \times (\phi 0 + x).$$

But the easiest method is this. To separate an expression into its $\left\{ \begin{matrix} \text{ultimate aggregants} \\ \text{prime factors} \end{matrix} \right\}$ take any $\left\{ \begin{matrix} \text{product} \\ \text{sum} \end{matrix} \right\}$ of all the different letters of the expression, each taken either positively or negatively (that is, with a dash over it). By means of the fundamental formulæ

$$\mathrm{X} \times \mathrm{Y} \prec \mathrm{Y} \prec \mathrm{Y} + \mathrm{Z},$$

examine whether the $\left\{ \begin{matrix} \text{product} \\ \text{sum} \end{matrix} \right\}$ taken is a $\left\{ \begin{matrix} \text{subject} \\ \text{predicate} \end{matrix} \right\}$ of every $\left\{ \begin{matrix} \text{factor} \\ \text{aggregant} \end{matrix} \right\}$ of the given expression. If so, it is a $\left\{ \begin{matrix} \text{ultimate aggregant} \\ \text{prime factor} \end{matrix} \right\}$ of that expression; otherwise not. Proceed in this way until as many $\left\{ \begin{matrix} \text{ultimate aggregants} \\ \text{prime factors} \end{matrix} \right\}$ have been found as the expression possesses. This number is found in the case of a $\left\{ \begin{matrix} \text{product of sums} \\ \text{sum of products} \end{matrix} \right\}$ of letters, as follows. Let m be the number of *different* letters in the expression (a letter and its negative not being considered different); let n be the total number of letters whether the same or different, and let p be the number of $\left\{ \begin{matrix} \text{factors} \\ \text{terms} \end{matrix} \right\}$. Then the number of $\left\{ \begin{matrix} \text{ultimate aggregants} \\ \text{prime factors} \end{matrix} \right\}$ is

$$2^m + n - mp - p.$$

For example, let it be required to separate $x + (y \times z)$ into its prime factors. Here $m = 3$, $n = 3$, $p = 2$. Hence the number of factors is three. Trying $x + y + z$, we have

$$x \prec x + y + z \qquad y \times z \prec x + y + z,$$

so that this is a factor. Trying $x + y + \bar{z}$, we have

$$x \prec x + y + \bar{z} \qquad y \times z \prec x + y + \bar{z},$$

so that this is also a factor. It is, also, obvious that $x + \bar{y} + z$ is the third factor. Accordingly,

$$x + (y \times z) = (x + y + z) \times (x + y + \bar{z}) \times (x + \bar{y} + z).$$

Again, let us develop the expression

$$(\bar{a} + b + c) \times (a + \bar{b} + \bar{c}) \times (a + b + c).$$

Here $m = 3$, $n = 9$, $p = 3$; so that the number of ultimate aggregants is five.

Of the eight possible products of three letters, then, only three are excluded, namely: $(a \times \bar{b} \times \bar{c})$, $(\bar{a} \times b \times c)$ and $(\bar{a} \times \bar{b} \times \bar{c})$. We have, then,

$$(\bar{a} + b + c) \times (a + \bar{b} + \bar{c}) \times (a + b + c) =$$
$$(a \times b \times c) + (a \times b \times \bar{c}) + (a \times \bar{b} \times c) + (\bar{a} \times b \times \bar{c}) + (\bar{a} \times \bar{b} \times c).$$

Third Process. Separate all complex propositions into simple ones by means of the following formulæ from the definitions of $+$ and $\times$:

$$\begin{aligned}
(\mathrm{X} + \mathrm{Y} \prec \mathrm{Z}) &= (\mathrm{X} \prec \mathrm{Z}) \times (\mathrm{Y} \prec \mathrm{Z}) \\
(\mathrm{X} \prec \mathrm{Y} \times \mathrm{Z}) &= (\mathrm{X} \prec \mathrm{Y}) \times (\mathrm{X} \prec \mathrm{Z}) \\
(\mathrm{X} + \mathrm{Y} \,\overline{\prec}\, \mathrm{Z}) &= (\mathrm{X} \,\overline{\prec}\, \mathrm{Z}) + (\mathrm{Y} \,\overline{\prec}\, \mathrm{Z}) \\
(\mathrm{X} \,\overline{\prec}\, \mathrm{Y} \times \mathrm{Z}) &= (\mathrm{X} \,\overline{\prec}\, \mathrm{Y}) + (\mathrm{X} \,\overline{\prec}\, \mathrm{Z}).
\end{aligned}$$

In practice, the first three operations will generally be performed off-hand in writing down the premises.

Fourth Process. If we have given two propositions, one of one of the forms

$$a \prec b + x \qquad\qquad a \times \bar{x} \prec b,$$

and the other of one of the forms

$$c \prec d + \bar{x} \qquad\qquad c \times x \prec d,$$

we may, by the transitiveness of the copula, eliminate x, and so obtain

$$a \times c \prec b + d.$$

Fifth Process. We may transpose any term from subject to predicate or the reverse, by changing it from positive to negative or the reverse, and at the same time its mode of connection from addition to multiplication or the reverse. Thus,

$$(x \times y \prec z) = (x \prec \bar{y} + z).$$

We may, in this way, obtain all the subjects and predicates of any letter; or we may bring all the letters into the subject, leaving the predicate 0, or all into the predicate, leaving the subject ∞.

Sixth Process. Any number of propositions having a common $\left\{\begin{matrix}\text{subject}\\ \text{predicate}\end{matrix}\right\}$ are, taken together, equivalent to their $\left\{\begin{matrix}\text{product}\\ \text{sum}\end{matrix}\right\}$.

As an example of this method, we may consider a well-known problem given by Boole. The data are

$$\begin{gathered}
\bar{x} \times \bar{z} \prec v \times (y \times \bar{w} + \bar{y} \times w) \\
\bar{v} \times x \times w \prec (y \times z) + (\bar{y} \times \bar{z}) \\
(x \times y) + (v \times x \times \bar{y}) = (z \times \bar{w}) + (\bar{z} \times w).
\end{gathered}$$

The quæsita are: first, to find those predicates of x which involve only y, z, and w; second, to find any relations which may be implied between y, z, w; third, to find the predicates of y; fourth, to find any relation which may be implied between x, z, and w. By the first three processes, mentally performed, we resolve the premises as follows: the first into

$$\bar{x} \times \bar{z} \prec v$$
$$\bar{x} \times \bar{z} \prec y + w$$
$$\bar{x} \times \bar{z} \prec \bar{y} + \bar{w};$$

the second into

$$\bar{v} \times x \times w \prec y + \bar{z}$$
$$\bar{v} \times x \times w \prec \bar{y} + z;$$

the third into

$$x \times y \prec z + w$$
$$x \times y \prec \bar{z} + \bar{w}$$
$$v \times x \times \bar{y} \prec z + w$$
$$v \times x \times \bar{y} \prec \bar{z} + \bar{w}$$
$$z \times \bar{w} \prec x$$
$$\bar{z} + w \prec v + y$$
$$z + \bar{w} \prec x$$
$$\bar{z} + w \prec v + y.$$

We must first eliminate v, about which we want to know nothing. We have, on the one hand, the propositions

$$v \times x \times \bar{y} \prec z + w$$
$$v \times x \times \bar{y} \prec \bar{z} + \bar{w};$$

and, on the other, the propositions

$$\bar{x} \times \bar{z} \prec v$$
$$\bar{v} \times x \times w \prec y + \bar{z}$$
$$\bar{v} \times x \times w \prec \bar{y} + z$$
$$z \times \bar{w} \prec v + y$$
$$\bar{z} \times w \prec v + y.$$

The conclusions from these propositions are obtained by taking one from each set, multiplying their subjects, adding their predicates, and omitting v. The result will be a merely empty proposition if the same letter in the same quality as to being positive or negative be found in the subject and in the predicate, or if it be found twice with opposite qualities either in the subject or in the predicate. Thus, it will be useless to combine the proposition $v \times x \times \bar{y} \prec z + w$ with any which contains $\bar{x}$, y, z, or w, in the subject. But all of the second set do this, so that nothing can be concluded from this proposition. So it will be

useless to combine $v \times x \times \bar{y} \prec \bar{z} + \bar{w}$ with any which contains $\bar{x}$, y, $\bar{z}$, $\bar{w}$ in the subject, or z in the predicate. This excludes every proposition of the second set except $\bar{v} \times x \times w \prec y + \bar{z}$, which, combined with the proposition under discussion, gives

$$x \times w \prec y + \bar{z} + \bar{w}$$

or

$$x \times w \prec y + \bar{z},$$

which is therefore to be used in place of all the premises containing v.

One of the other propositions, namely, $\bar{x} \times \bar{z} \prec \bar{y} + \bar{w}$ is obviously contained in another, namely: $\bar{z} \times w \prec x$. Rejecting it, our premises are reduced to six, namely:

$$\begin{aligned} \bar{x} \times \bar{z} &\prec y + w \\ x \times y &\prec z + w \\ x \times y &\prec \bar{z} + \bar{w} \\ z \times \bar{w} &\prec x \\ \bar{z} \times w &\prec x \\ x \times w &\prec y + \bar{z}. \end{aligned}$$

The second, third, and sixth of these give the predicates of x. Their product is

$$x \prec (\bar{y} + z + w) \times (\bar{y} + \bar{z} + \bar{w}) \times (y + \bar{z} + \bar{w})$$

or

$$x \prec y \times z \times \bar{w} + y \times \bar{z} \times w + \bar{y} \times z \times \bar{w} + \bar{y} \times \bar{z} \times w + \bar{y} \times \bar{z} \times \bar{w}$$

or

$$x \prec z \times \bar{w} + \bar{z} \times w + \bar{y} \times \bar{z} \times \bar{w}.$$

To find whether any relation between y, z, and w can be obtained by the elimination of x, we find the subjects of x by combining the first, fourth, and fifth premises. Thus we find

$$\bar{y} \times \bar{z} \times \bar{w} + z \times \bar{w} + \bar{z} \times w \prec x.$$

It is obvious that the conclusion from the last two propositions is a merely identical proposition, and therefore no independent relation is implied between y, z, and w.

To find the predicates of y we combine the second and third propositions. This gives

$$y \prec (\bar{x} + z + w) \times (\bar{x} + \bar{z} + \bar{w})$$

or

$$y \prec x \times z \times \bar{w} + x \times \bar{z} \times w + \bar{x}.$$

Two relations between x, z, and w are given in the premises, namely: $z \times \bar{w} \prec x$ and $\bar{z} \times w \prec x$. To find whether any other is implied, we eliminate y between the above proposition and the first and sixth premises. This gives

$$\begin{aligned} \bar{x} \times \bar{z} &\prec x \times z \times \bar{w} + w + \bar{x} \\ x \times w &\prec x \times z \times \bar{w} + \bar{x} + \bar{z}. \end{aligned}$$

The first conclusion is empty. The second is equivalent to $x \times w \prec \bar{z}$, which is a third relation between x, z, and w.

Everything implied in the premises in regard to the relations of x, y, z, w may be summed up in the proposition

$$\infty \prec x + z \times w + y \times \bar{z} \times \bar{w}.$$

CHAPTER III. — THE LOGIC OF RELATIVES.

§ 1. *Individual and Simple Terms.*

Just as we had to begin the study of Logical Addition and Multiplication by considering ∞ and 0, terms which might have been introduced under the Algebra of the Copula, being defined in terms of the copula only, without the use of $+$ or $\times$, but which had not been there introduced, because they had no application there, so we have to begin the study of relatives by considering the doctrine of individuals and simples, — a doctrine which makes use only of the conceptions of non-relative logic, but which is wholly without use in that part of the subject, while it is the very foundation of the conception of a relative, and the basis of the method of working with the algebra of relatives.

The germ of the correct theory of individuals and simples is to be found in Kant's *Critic of the Pure Reason, Appendix to the Transcendental Dialectic*, where he lays it down as a regulative principle, that, if

$$a \prec b \qquad\qquad b \mathrel{\overline{\prec}} a,$$

then it is always possible to find such a term x, that

$$\begin{array}{ll} a \prec x & \qquad x \prec b \\ x \mathrel{\overline{\prec}} a & \qquad b \mathrel{\overline{\prec}} x. \end{array}$$

Kant's distinction of regulative and constitutive principles is unsound, but this *law of continuity*, as he calls it, must be accepted as a fact. The proof of it, which I have given elsewhere, depends on the continuity of space, time, and the intensities of the qualities which enter into the definition of any term. If, for instance, we say that Europe, Asia, Africa and North America are continents, but not all the continents, there remains over only South America. But we may distinguish between South America as it now exists and South America in former geological times; we may, therefore, take x as including Europe, Asia, Africa,

North America, and South America as it exists now, and every x is a continent, but not every continent is x.

Just as in mathematics we speak of infinitesimals and infinites, which are fictitious limits of continuous quantity, and every statement involving these expressions has its interpretation in the doctrine of limits, so in logic we may define an *individual*, A, as such a term that

$$\mathrm{A} \mathrel{\overline{-\!\!<}} 0,$$

but such that if

$$x < \mathrm{A}$$

then $$x \mathrel{-\!\!<} 0.$$

And in the same way, we may define the *simple*, a, as such a term that

$$1 \mathrel{\overline{-\!\!<}} a,$$

but such that if

$$a < x$$

then $$1 \mathrel{-\!\!<} x.$$

The individual and the simple, as here defined, are ideal limits, and every statement about either is to be interpreted by the doctrine of limits.

Every term may be conceived as a limitless logical sum of individuals, or as a limitless logical product of simples; thus,

$$a = \mathrm{A}_1 + \mathrm{A}_2 + \mathrm{A}_3 + \mathrm{A}_4 + \mathrm{A}_5 + \text{etc.}$$
$$\bar{a} = \bar{\mathrm{A}}_1 \times \bar{\mathrm{A}}_2 \times \bar{\mathrm{A}}_3 \times \bar{\mathrm{A}}_4 \times \bar{\mathrm{A}}_5 \times \text{etc.}$$

It will be seen that a simple is the negative of an individual.

§ 2. *Relatives.*

A *relative* is a term whose definition describes what sort of a system of objects that is whose first member (which is termed the *relate*) is denoted by the term; and names for the other members of the system (which are termed the *correlates*) are usually appended to limit the denotation still further. In these systems the order of the members is essential; so that (A, B, C) and (A, C, B) are different systems. As an example of a relative, take 'buyer of — for — from'; we may append to this three correlates, thus, 'buyer of every horse of a certain description in the market for a good price from its owner.'

A relative of only one correlate, so that the system it supposes is a pair, may be called a *dual* relative; a relative of more than one correlate may be called *plural*. A non-relative term may be called a term of *singular reference*.

Every relative, like every term of singular reference, is general; its defini-

tion describes a system in general terms; and, as general, it may be conceived either as a logical sum of individual relatives, or as a logical product of simple relatives.* An individual relative refers to a system all the members of which are individual. The expressions

$$(A : B) \qquad (A : B : C)$$

may denote individual relatives. Taking dual individual relatives, for instance, we may arrange them all in an infinite block, thus,

A : A	A : B	A : C	A : D	A : E	etc.
B : A	B : B	B : C	B : D	B : E	etc.
C : A	C : B	C : C	C : D	C : E	etc.
D : A	D : B	D : C	D : D	D : E	etc.
E : A	E : B	E : C	E : D	E : E	etc.
etc.	etc.	etc.	etc.	etc.	

In the same way, triple individual relatives may be arranged in a cube, and so forth. The logical sum of all the relatives in this infinite block will be the relative universe, ∞, where

$$x \prec \infty,$$

whatever dual relative x may be. It is needless to distinguish the dual universe, the triple universe, etc., because, by adding a perfectly indefinite additional member to the system, a dual relative may be converted into a triple relative, etc. Thus, for *lover of a woman*, we may write *lover of a woman coexisting with anything*. In the same way, a term of single reference is equivalent to a relative with an indefinite correlate; thus, *woman* is equivalent to *woman coexisting with anything*. Thus, we shall have

$$A = A : A + A : B + A : C + A : D + A : E + \text{etc.}$$
$$A : B = A : B : A + A : B : B + A : B : C + A : B : D + \text{etc.}$$

From the definition of a simple term given in the last section, it follows that every simple relative is the negative of an individual term. But while in non-relative logic negation only divides the universe into two parts, in relative logic the same operation divides the universe into 2^n parts, where n is the number of objects in the system which the relative supposes; thus,

$$\infty = A + \bar{A}$$
$$\infty = A : B + \bar{A} : B + A : \bar{B} + \bar{A} : \bar{B}$$

* In my *Logic of Relatives*, 1870, I have used this expression to designate what I now call *dual relatives*.

$$\infty = (A : B : C) + (\bar{A} : B : C) + (A : \bar{B} : C) + (A : B : \bar{C})$$
$$+ (\bar{A} : \bar{B} : \bar{C}) + (A : \bar{B} : \bar{C}) + (\bar{A} : B : \bar{C}) + (\bar{A} : \bar{B} : C).$$

Here, we have

$$A = A : B + A : \bar{B}; \quad \bar{A} = \bar{A} : B + \bar{A} : \bar{B};$$
$$A : B = A : B : C + A : B : \bar{C}; \quad A : \bar{B} = A : \bar{B} : C + A : \bar{B} : \bar{C};$$
$$\bar{A} : B = \bar{A} : B : C + \bar{A} : B : \bar{C}; \quad \bar{A} : \bar{B} = \bar{A} : \bar{B} : C + \bar{A} : \bar{B} : \bar{C}.$$

It will be seen that a term which is individual when considered as n-fold is not so when considered as more than n-fold; but an n-fold term when made $(m+n)$-fold, is individual as to n members of the system, and indefinite as to m members.

Instead of considering the system of a relative as consisting of non-relative individuals, we may conceive of it as consisting of relative individuals. Thus, since

$$A = A : A + A : B + A : C + A : D + \text{etc.},$$

we have

$$A : B = (A : A) : B + (A : B) : B + (A : C) : B + (A : D) : B + \text{etc.}$$

But

$$B = B : A + B : B + B : C + B : D + \text{etc.};$$

so that

$$A : B = A : (B : A) + A : (B : B) + A : (B : C) + A : (B : D) + \text{etc.}$$

Here we have evidently

$$(A : C) : B = A : (B : C).$$

In the same way we find

$$\begin{aligned}(A : D) : (B : C) &= (A : C) : (B : D)\\ &= A : [(B : D) : C] = A : [B : (C : D)]\\ &= [A : (C : D)] : B = [(A : D) : C] : B.\end{aligned}$$

§ 3. *Relatives connected by Transposition of Relate and Correlate.*

Connected with every dual relative, as

$$l = \Sigma (A : B) = \Pi (\alpha : \beta),$$

is another which is called its *converse*,

$$k\text{-}l = \Sigma (B : A) = \Pi (\beta : \alpha),$$

in which the relate and correlate are transposed. The converse, k, is itself a relative, being

$$k = \Sigma [(A : B) : (B : A)];$$

that is, it is the first of any pair which embraces two individual dual relatives, each of which is the converse of the other. The converse of the converse is the relation itself, thus

$$k\text{-}k\text{-}l = l,$$

or say

$$kk = 1.$$

We have also

$$k\text{-}\bar{l} = \overline{k\text{-}l}$$
$$k\Sigma = \Sigma k$$
$$k\Pi = \Pi k.$$

In the case of triple relatives there are five transpositions possible. Thus, if

$$b = \Sigma\,[(\mathrm{A} : \mathrm{B}) : \mathrm{C}] = \Sigma\,[\mathrm{A} : (\mathrm{C} : \mathrm{B})],$$

we may write

$$\begin{aligned}
\mathrm{I}b &= \Sigma\,[(\mathrm{B} : \mathrm{A}) : \mathrm{C}] = \Sigma\,[\mathrm{B} : (\mathrm{C} : \mathrm{A})]\\
\mathrm{J}b &= \Sigma\,[\mathrm{A} : (\mathrm{B} : \mathrm{C})] = \Sigma\,[(\mathrm{A} : \mathrm{C}) : \mathrm{B}]\\
\mathrm{K}b &= \Sigma\,[\mathrm{C} : (\mathrm{A} : \mathrm{B})] = \Sigma\,[(\mathrm{C} : \mathrm{B}) : \mathrm{A}]\\
\mathrm{L}b &= \Sigma\,[(\mathrm{C} : \mathrm{A}) : \mathrm{B}] = \Sigma\,[\mathrm{C} : (\mathrm{B} : \mathrm{A})]\\
\mathrm{M}b &= \Sigma\,[\mathrm{B} : (\mathrm{A} : \mathrm{C})] = \Sigma\,[(\mathrm{B} : \mathrm{C}) : \mathrm{A}].
\end{aligned}$$

Here we have

$$\begin{aligned}
&\mathrm{LM} = \mathrm{ML} = 1\\
&\mathrm{II} = \mathrm{JJ} = \mathrm{KK} = 1\\
&\mathrm{IJ} = \mathrm{JK} = \mathrm{KI} = \mathrm{L}\\
&\mathrm{JI} = \mathrm{KJ} = \mathrm{IK} = \mathrm{M}\\
&\mathrm{IL} = \mathrm{MI} = \mathrm{J} = \mathrm{KM} = \mathrm{LK}\\
&\mathrm{JL} = \mathrm{MJ} = \mathrm{K} = \mathrm{IM} = \mathrm{LI}\\
&\mathrm{KL} = \mathrm{MK} = \mathrm{I} = \mathrm{JM} = \mathrm{LJ}.
\end{aligned}$$

If we write $a : b$ to express the operation of putting A in place of B in the original relative

$$b = \Sigma\,[(\mathrm{A} : \mathrm{B}) : \mathrm{C}] = \Sigma\,[\mathrm{A} : (\mathrm{C} : \mathrm{B})],$$

then we have

$$\begin{aligned}
\mathrm{I} &= a : b + b : a + c : c\\
\mathrm{J} &= a : a + b : c + c : b\\
\mathrm{K} &= a : c + b : b + c : a\\
\mathrm{L} &= a : b + b : c + c : a\\
\mathrm{M} &= a : c + b : a + c : b\\
1 &= a : a + b : b + c : c.
\end{aligned}$$

Then we have

$$\mathrm{I} + \mathrm{J} + \mathrm{K} = 1 + \mathrm{L} + \mathrm{M},$$

which does not imply

$$(\mathrm{I} + \mathrm{J} + \mathrm{K})\,l = (1 + \mathrm{L} + \mathrm{M})\,l.$$

In a similar way the n-fold relative will have $(n\,! - 1)$ transposition-functions.

§ 4. *Classification of Relatives.*

Individual relatives are of one or other of the two forms

$$A:A \qquad\qquad A:B,$$

and simple relatives are negatives of one or other of these two forms.

The forms of general relatives are of infinite variety, but the following may be particularly noticed.

Relatives may be divided into those all whose individual aggregants are of the form A : A and those which contain individuals of the form A : B. The former may be called *concurrents*, the latter *opponents*. Concurrents express a mere agreement among objects. Such, for instance, is the relative '*man that is* —,' and a similar relative may be formed from any term of singular reference. We may denote such a relative by the symbol for the term of singular reference with a comma after it; thus $(m,)$ will denote 'man that is —' if (m) denotes 'man.' In the same way a comma affixed to an n-fold relative will convert it into an $(n+1)$-fold relative. Thus, (l) being 'lover of —,' $(l,)$ will be 'lover that is — of —.'

The negative of a concurrent relative will be one each of whose simple components is of the form $\overline{A:A}$, and the negative of an opponent relative will be one which has components of the form $\overline{A:B}$.

We may also divide relatives into those which contain individual aggregants of the form A : A and those which contain only aggregants of the form A : B. The former may be called *self-relatives*, the latter *alio-relatives*. We also have negatives of self-relatives and negatives of alio-relatives.

These different classes have the following relations. Every negative of a concurrent and every alio-relative is both an opponent and the negative of a self-relative. Every concurrent and every negative of an alio-relative is both a self-relative and the negative of an opponent. There is only one relative which is both a concurrent and the negative of an alio-relative; this is 'identical with —.' There is only one relative which is at once an alio-relative and the negative of a concurrent; this is the negative of the last, namely, 'other than —.' The following pairs of classes are mutually exclusive, and divide all relatives between them:

Alio-relatives and self-relatives,
Concurrents and opponents,
Negatives of alio-relatives and negatives of self-relatives,
Negatives of concurrents and negatives of opponents.

No relative can be at once either an alio-relative or the negative of a concurrent, and at the same time either a concurrent or the negative of an alio-relative.

We may append to the symbol of any relative a semicolon to convert it into an alio-relative of a higher order. Thus $(l;)$ will denote a 'lover of—that is not—.'

This completes the classification of dual relatives founded on the difference of the fundamental forms A : A and A : B. Similar considerations applied to triple relatives would give rise to a highly complicated development, inasmuch as here we have no less than five fundamental forms of individuals, namely,

$$(A:A):A \quad (A:A):B \quad (A:B):A \quad (B:A):A \quad (A:B):C.$$

The number of individual forms for the $(n+2)$-fold relative is

$$2+(2^n-1)\,.\,3+\frac{1}{2!}\left\{(3^n-1)-2\,(2^n-1)\right\}.\,4+\frac{1}{3!}\left\{(4^n-1)-3\,(3^n-1)\right.$$
$$\left.+3\,(2^n-1)\right\}.\,5+\frac{1}{4!}\left\{(5^n-1)-4\,(4^n-1)+6\,(3^n-1)-4\,(2^n-1)\right\}.\,6$$
$$+\frac{1}{5!}\left\{(6^n-1)-5\,(5^n-1)+10\,(4^n-1)-10\,(3^n-1)+5\,(2^n-1)\right\}.\,7+\text{etc.}$$

If this number be called fn, we have

$$\Delta^n f0 = f(n-1)$$
$$f0 = 1.$$

The form of calculation is

1					
2	1				
5	3	2			
15	10	7	5		
52	37	27	20	15	
203	151	114	87	67	52

where the diagonal line is copied number by number from the vertical line, as fast as the latter is computed.

Relatives may also be classified according to the general amount of filling up of the above-mentioned block, cube, etc. they present. In the first place, we have such relatives in whose block, cube, etc. every line in a certain direction in which there is a single individual is completely filled up. Such relatives may be called *complete in regard to* the relate, or first, second, third, etc. correlate. The dual relatives which are equivalent to terms of singular reference are complete as to their correlate.

A relative may be incomplete with reference to a certain correlate or to its relate, and yet every individual of the universe may in some way enter into that correlate or relate. Such a relative may be called *unlimited* in reference to the correlate or relate in question. Thus, the relative

$$A:A+A:B+C:C+C:D+E:E+E:F+G:G+G:H+\text{etc.}$$

is unlimited as to its correlate. The negative of an unlimited relative will be unlimited unless the relative has as an integrant a relative which is complete with regard to every other relate and correlate than that with reference to which the given relative is unlimited.

A totally unlimited relative is one which is unlimited in reference to the relate and all the correlates. A totally unlimited relative in which each letter enters only once into the relate and once into the correlate is termed a *substitution.*

Certain classes of relatives are characterized by the occurrence or non-occurrence of certain individual aggregants related in a definite way to others which occur. A set of individual dual relatives each of which has for its relate the correlate of the last, the last of all being considered as preceding the first of all, may be called a *cycle.* If there are n individuals in the cycle it may be called a cycle of the n^{th} order. For instance,

$$A:B \quad B:C \quad C:D \quad D:E \quad E:A$$

may be called the cycle of the fifth order. Now, if a certain relative be such that of any cycle of the n^{th} order of which it contains any m terms, it also contains the remaining $(n-m)$ terms, it may be called a cyclic relative of the n^{th} order and m^{th} degree. If. on the other hand, of any cycle of the n^{th} order of which it contains m terms the remaining $(n-m)$ are wanting, the relative may be called an anticyclic relative of the n^{th} order and m^{th} degree.

A cyclic relative of the first order and first degree contains all individual components of the form $A:A$. A cyclic relative of the second order and first degree is called an *equiparant* in opposition to a *disquiparant.*

A highly important class of relatives is that of *transitives;* that is to say, those which for every two individual terms of the forms $(A:B)$ and $(B:C)$ also possess a term of the form $(A:C)$.

§ 5. *The Composition of Relatives.*

Suppose two relatives either individual or simple, and having the relate or correlate of the first identical with the relate or correlate of the second or of

its negative. This pair of relatives will then be of one or other of sixteen forms, viz.:

$$
\begin{array}{llll}
(A:B)\ (B:C) & (\overline{A:B})\ (B:C) & (A:B)\ (\overline{B:C}) & (\overline{A:B})\ (\overline{B:C}) \\
(A:B)\ (C:B) & (\overline{A:B})\ (C:B) & (A:B)\ (\overline{C:B}) & (\overline{A:B})\ (\overline{C:B}) \\
(B:A)\ (B:C) & (\overline{B:A})\ (B:C) & (B:A)\ (\overline{B:C}) & (\overline{B:A})\ (\overline{B:C}) \\
(B:A)\ (C:B) & (\overline{B:A})\ (C:B) & (B:A)\ (\overline{C:B}) & (\overline{B:A})\ (\overline{C:B}).
\end{array}
$$

Now we may conceive an operation upon any one of these sixteen pairs of relatives of such a nature that it will produce one or other of the four forms $(A:C)$, $(\overline{A:C})$, $(C:A)$, $(\overline{C:A})$. Thus, we shall have sixty-four operations in all.

We may symbolize them as follows: Let

$$A:B\ (|||)\ B:C = A:C;$$

that is, let $(|||)$ signify such an operation that this formula necessarily holds. The three lines in the sign of this operation are to refer respectively to the first relative operated upon, the second relative operated upon, and to the result. When either of these lines is replaced by a hyphen $(-)$, let the operation signified be such that the negative of the corresponding relative must be substituted in the above formula. Thus,

$$\overline{A:B}\ (-||)\ B:C = A:C.$$

In the same way, let a semicircle $(\smile)$ signify that the converse of the corresponding relative is to be taken. The hyphen and the semicircle may be used together. If, then, we write the symbol of a relative with a semicircle or curve over it to denote the converse of that relative, we shall have, for example,

$$\overset{\smile}{A:B}\ (\smile||)\ B:C = A:C.$$

Then any combination of the relatives a and e, in this order, is equivalent to others formed from this by making any of the following changes:

First. Putting a straight or curved mark over a and changing the first mark of the sign of operation in the corresponding way; that is,

for $\breve{a}$, from $|$ to $\smile$ or from $-$ to $\underline{\smile}$ or conversely,
for $\bar{a}$, from $|$ to $-$ or from $\smile$ to $\underline{\smile}$ or conversely,
for $\breve{\bar{a}}$, from $|$ to $\underline{\smile}$ or from $-$ to $\smile$ or conversely.

Second. Making similar simultaneous modifications of e and of the second mark.

Third. Changing the third mark from $|$ to $-$ or from $\smile$ to $\underline{\smile}$ or conversely, and simultaneously writing the mark of negation over the whole expression.

Fourth. Changing the third mark from | to ⌣ or from – to ⩡ or conversely, and interchanging a and e and also the first and second marks.

We have thus far defined the effect of the sixty-four operations when certain members of the individual relatives operated upon are identical. When these members are not identical, we may suppose either that the operation ||| produces either the first or second relative or 0. We cannot suppose that it produces ∞ for a reason which will appear further on. Let us elect the formula

$$\mathrm{A}:\mathrm{B}\;(|||)\;\mathrm{C}:\mathrm{D} = 0.$$

The other excluded operations will be included in a certain manner, as we shall see below. From this formula, by means of the rules of equivalence, it will follow that all operations in whose symbol there is no hyphen in the third place will also give 0 in like circumstances, while all others will give $\bar{0}$ or ∞.

We have thus far only defined the effect of the sixty-four operations upon individual or simple terms. To extend the definitions to other cases, let us suppose first that the rules of equivalence are generally valid, and second, that

$$\text{If } a \mathrel{-\!\!<} b \text{ and } c \mathrel{-\!\!<} d \text{ then } a\;(|||)\;c \mathrel{-\!\!<} b\;(|||)\;d$$

or

$$(a \mathrel{-\!\!<} b) \times (c \mathrel{-\!\!<} d) \mathrel{-\!\!<} \{a\;(|||)\;c \mathrel{-\!\!<} b\;(|||)\;d\}.$$

Then, this rule will hold good in all operations in whose symbols the first and second places agree with the third in respect to having or not having hyphens. For operations, in whose symbols the $\left\{\begin{matrix}\text{first}\\ \text{second}\end{matrix}\right\}$ mark disagrees with the third in this respect we must write $\left\{\begin{matrix} b \mathrel{-\!\!<} a \\ d \mathrel{-\!\!<} c\end{matrix}\right\}$ instead of $\left\{\begin{matrix} a \mathrel{-\!\!<} b \\ c \mathrel{-\!\!<} d\end{matrix}\right\}$ in this rule. Thus, the sixty-four operations are divisible into four classes according to which one of the four rules so produced they follow.

It now appears that only the hyphens and not the curved marks are of significance in reference to the rule which an operation follows. Let us accordingly reject all operations whose symbols contain curved marks, and there remain only eight. For these eight the following formulæ hold:

$$\begin{array}{ll}
\mathrm{A}:\mathrm{B}\;(|||)\;\mathrm{B}:\mathrm{C} = \mathrm{A}:\mathrm{C} & \mathrm{A}:\mathrm{B}\;(||-)\;\mathrm{B}:\mathrm{C} = \overline{\mathrm{A}:\mathrm{C}} \\
\overline{\mathrm{A}:\mathrm{B}}\;(-||)\;\mathrm{B}:\mathrm{C} = \mathrm{A}:\mathrm{C} & \overline{\mathrm{A}:\mathrm{B}}\;(-|-)\;\mathrm{B}:\mathrm{C} \doteq \overline{\mathrm{A}:\mathrm{C}} \\
\mathrm{A}:\mathrm{B}\;(|-|)\;\overline{\mathrm{B}:\mathrm{C}} = \mathrm{A}:\mathrm{C} & \mathrm{A}:\mathrm{B}\;(|--)\;\overline{\mathrm{B}:\mathrm{C}} = \overline{\mathrm{A}:\mathrm{C}} \\
\overline{\mathrm{A}:\mathrm{B}}\;(--|)\;\overline{\mathrm{B}:\mathrm{C}} = \mathrm{A}:\mathrm{C} & \overline{\mathrm{A}:\mathrm{B}}\;(---)\;\overline{\mathrm{B}:\mathrm{C}} = \overline{\mathrm{A}:\mathrm{C}}
\end{array}$$

$$\begin{array}{ll} \mathrm{A:B}\ (|||)\ \mathrm{C:D}=0 & \mathrm{A:B}\ (||-)\ \mathrm{C:D}=\infty \\ \overline{\mathrm{A:B}}\ (-||)\ \mathrm{C:D}=0 & \overline{\mathrm{A:B}}\ (-|-)\ \mathrm{C:D}=\infty \\ \mathrm{A:B}\ (|-|)\ \overline{\mathrm{C:D}}=0 & \mathrm{A:B}\ (|--)\ \overline{\mathrm{C:D}}=\infty \\ \overline{\mathrm{A:B}}\ (--|)\ \overline{\mathrm{C:D}}=0 & \overline{\mathrm{A:B}}\ (---)\ \overline{\mathrm{C:D}}=\infty \end{array}$$

$$(a \prec b) \times (c \prec d) \prec \{a\ (|||)\ c \prec b\ (|||)\ d\}$$
$$(a \prec b) \times (c \prec d) \prec \{a\ (---)\ c \prec b\ (---)\ d\}$$

$$(b \prec a) \times (c \prec d) \prec \{a\ (-||)\ c \prec b\ (-||)\ d\}$$
$$(b \prec a) \times (c \prec d) \prec \{a\ (|--)\ c \prec b\ (|--)\ d\}$$

$$(a \prec b) \times (d \prec c) \prec \{a\ (|-|)\ c \prec b\ (|-|)\ d\}$$
$$(a \prec b) \times (d \prec c) \prec \{a\ (-|-)\ c \prec b\ (-|-)\ d\}$$

$$(b \prec a) \times (d \prec c) \prec \{a\ (--|)\ c \prec b\ (--|)\ d\}$$
$$(b \prec a) \times (d \prec c) \prec \{a\ (||-)\ c \prec b\ (||-)\ d\}.$$

As it is inconvenient to consider so many as eight distinct operations, we may reject one-half of these so as to retain one under each of the four rules. We may reject all those whose symbols contain an odd number of hyphens (as being negative). We then retain four, to which we may assign the following names and symbols:

$a\ (|||)\ e = ae$ *Relative or external multiplication.*
$a\ (|--)\ e = {}^a e$ *Regressive involution.*
$a\ (-|-)\ e = a^e$ *Progressive involution.*
$a\ (--|)\ e = a \circ e$ *Transaddition.**

We have then the following table of equivalents, negatives, and converses:†

x	$\bar{x}$	$\breve{x}$	$\breve{\bar{x}}$
$ae = \bar{a}\circ\bar{e}$	$\bar{a}^e = {}^a\bar{e}$	$\breve{e}\breve{a} = \breve{\bar{e}}\circ\breve{\bar{a}}$	$\breve{\bar{e}}^{\breve{a}} = {}^{\breve{e}}\breve{\bar{a}}$
$a^e = {}^{\bar{a}}\bar{e}$	$\bar{a}e = a\circ\bar{e}$	${}^{\breve{e}}\breve{a} = \breve{\bar{e}}^{\breve{\bar{a}}}$	$\breve{\bar{e}}\breve{\bar{a}} = \breve{e}\circ\breve{a}$
${}^a e = \bar{a}^{\bar{e}}$	$a\bar{e} = \bar{a}\circ e$	$\breve{e}^{\breve{a}} = {}^{\breve{\bar{e}}}\breve{\bar{a}}$	$\breve{\bar{e}}\breve{a} = \breve{e}\circ\breve{\bar{a}}$
$a\circ e = \bar{a}\bar{e}$	$a^{\bar{e}} = {}^{\bar{a}}e$	$\breve{e}\circ\breve{a} = \breve{\bar{e}}\breve{\bar{a}}$	$\breve{e}^{\breve{\bar{a}}} = {}^{\breve{\bar{e}}}\breve{a}$

---

* The first three of these were studied by De Morgan (*On the Syllogism*, No. IV.); the last is new. The above names for the first three (except the adjective *internal* suggested by Grassmann's operation) are given in my *Logic of Relatives.*

† A similar table is given by De Morgan. Of course, it lacks the symmetry of this, because he had not the fourth operation.

If l denote 'lover' and s 'servant,' then

ls denotes whatever is lover of a servant of –,

l^s whatever is lover of every servant of –,

ls whatever is in every way (in which it loves at all) lover of a servant,

$l \circ s$ whatever is not a non-lover only of a servant of –

or whatever is not a lover of everything but servants of –

or whatever is some way a non-lover of some thing besides servants of –.

§ 6. *Methods in the Algebra of Relatives.*

The universal method in this algebra is the method of limits. For certain letters are to be substituted an infinite sum of individuals or product of simples; whereupon certain transformations become possible which could not otherwise be effected.

The following theorems are indispensable for the application of this method:

1st. $$l^{\mathrm{A:B}} = l(\mathrm{A:B}) + k\bar{\mathrm{B}}.$$

Since $\bar{\mathrm{B}}$ is equivalent to the relative term which comprises all individual relatives whose relates are not B, so $k\bar{\mathrm{B}}$ may be conveniently used, as it is here, to express the aggregate of all individual relatives whose correlate is $\bar{\mathrm{B}}$. To prove this proposition, we observe that

$$l^{\mathrm{A:B}} = \overline{\bar{l}(\mathrm{A:B})}.$$

Now $\bar{l}(\mathrm{A:B})$ contains only individual relatives whose correlate is B, and of these it contains those which are not included in $l(\mathrm{A:B})$. Hence the negative of $\bar{l}(\mathrm{A:B})$ contains all individual relatives whose correlates are not B, together with all contained in $l(\mathrm{A:B})$. Q. E. D.

2d. $${}^{\mathrm{A:B}}l = (\mathrm{A:B})l + \bar{\mathrm{A}}.$$

Here $\bar{\mathrm{A}}$ is used to denote the aggregate of all individual relatives whose relates are not A. This proposition is proved like the last.

3d. $$\overline{\mathrm{A:B}}^{l} = (\mathrm{A:B})\,\bar{l} + \bar{\mathrm{A}}.$$

This is evident from the second proposition, because

$$\overline{\mathrm{A:B}}^{l} = {}^{(\mathrm{A:B})}\bar{l}.$$

4th. $${}^{l}\overline{\mathrm{A:B}} = \bar{l}(\mathrm{A:B}) + k\bar{\mathrm{B}}.$$

Another method of working with the algebra is by means of negations. This becomes quite indispensable when the operations are defined by negations, as in this paper.

To illustrate the use of these methods. let us investigate the relations of ${}^{l}b$ and l^{b} to lb when l and b are totally unlimited relatives.

Write $$l = \Sigma_i (L_i : M_i) \qquad b = \Sigma_j (B_j : C_j).$$

Then, by the rules of the last section,

$${}^{l}b \prec {}^{L:M}b \qquad l^{b} \prec l^{B \cdot C};$$

whence, by the second and third propositions above,

$${}^{l}b \prec (L_i : M_i)\, b + \bar{L}_i \qquad l^{b} \prec l(B_j : C_j) + k\bar{B}_j.$$

But by the first rule of the last section

$$(L_i : M_i)\, b \prec lb \qquad l(B_j : C_j) \prec lb;$$

hence,

$${}^{l}b \prec lb + \bar{L}_i \qquad l^{b} \prec lb + k\bar{B}_j.$$

There will be propositions like these for all the different values of i and j. Multiplying together all those of the several sets, we have

$${}^{l}b \prec lb + \Pi_i \bar{L}_i \qquad l^{b} \prec lb + \Pi_j k\bar{B}_j.$$

But

$$\Pi_i \bar{L}_i = \overline{\Sigma_i L_i} \qquad \Pi_j k\bar{B}_j = \overline{\Sigma_j k B_j},$$

and since the relatives are unlimited,

$$\Sigma_i L_i = \infty \qquad \Sigma_j k B_j = \infty$$
$$\overline{\Sigma_i L_i} = 0 \qquad \overline{\Sigma_j k B_j} = 0;$$

hence

$${}^{l}b \prec lb \qquad l^{b} \prec lb.$$

In the same way it is easy to show that, if the negatives of l and b are totally unlimited,

$$l^{b} \prec l \circ b \qquad {}^{l}b \prec l \circ b.$$

§ 7. *The General Formulæ for Relatives.*

The principal formulæ of this algebra may be divided into ***distribution formulæ*** and *association formulæ*. The distribution formulæ are those which give the equivalent of a relative compounded with a sum or product of two relatives in such terms as to separate the latter two relatives. The association formulæ are those which give the equivalent of a relative A compounded with a compound of B and C in terms of a compound of A and B compounded with C.

I. DISTRIBUTION FORMULÆ.

1. Affirmative.

i. *Simple Formulæ.*

$$(a+b)\,c = ac + bc \qquad a\,(b+c) = ab + ac$$

$$(a \times b)^c = a^c \times b^c \qquad a^{b+c} = a^b \times a^c$$

$${}^{a+b}c = {}^{a}c \times {}^{b}c \qquad {}^{a}(b \times c) = {}^{a}b \times {}^{a}c$$

$$(a \times b)\circ c = (a\circ c) + (b\circ c) \qquad a\circ(b \times c) = (a\circ b) + (a\circ c)$$

ii. *Developments.*

$$(a \times b)\,c = \Pi_p \{\, a\,(c \times p) + b\,(c \times \bar{p}) \,\} \qquad a\,(b \times c) = \Pi_p \{\, (a \times p)\,b + (a \times \bar{p})\,c \}$$

$$(a+b)^c = \Sigma_p \{\, a^{c \times p} \times b^{c \times \bar{p}} \,\} \qquad a^{b \times c} = \Sigma_p \{\, (a+p)^b \times (a+\bar{p})^c \}$$

$${}^{(a \times b)}c = \Sigma_p \{\, {}^{a}(c+p) \times {}^{b}(c+\bar{p}) \,\} \qquad {}^{a}(b+c) = \Sigma_p \{\, {}^{a \times p}b \times {}^{a \times \bar{p}}c \}$$

$$(a+b)\circ c = \Pi_p \{ a\circ(c+p) + b\circ(c+\bar{p}) \} \qquad a\circ(b+c) = \Pi_p \{ (a+p)\circ b + (a+\bar{p})\circ c \}$$

2. Negative.

i. *Simple Formulæ.*

$$\overline{(a+b)\,c} = \overline{ac} \times \overline{bc} \qquad \overline{a\,(b+c)} = \overline{ab} + \overline{ac}$$

$$\overline{(a \times b)^c} = \overline{a^c} + \overline{b^c} \qquad \overline{a^{b+c}} = \overline{a^b} + \overline{a^c}$$

$$\overline{{}^{a+b}c} = \overline{{}^{a}c} + \overline{{}^{b}c} \qquad \overline{{}^{a}(b \times c)} = \overline{{}^{a}b} + \overline{{}^{a}c}$$

$$\overline{(a \times b)\circ c} = \overline{a\circ c} \times \overline{b\circ c} \qquad \overline{a\circ(b \times c)} = \overline{a\circ b} \times \overline{a\circ c}$$

ii. *Developments.*

$$\overline{(a \times b)\,c} = \Sigma_p \{ \overline{a\,(c \times p)} \times \overline{b\,(c \times \bar{p})} \} \qquad \overline{a\,(b \times c)} = \Sigma_p \{\, \overline{(a \times p)\,b} \times \overline{(a \times \bar{p})\,c} \}$$

$$\overline{(a+b)^c} = \Pi_p \{\, \overline{a^{c \times p}} + \overline{b^{c \times p}} \,\} \qquad \overline{a^{b \times c}} = \Pi_p \{\, \overline{(a+p)^b} + \overline{(a+\bar{p})^c} \}$$

$$\overline{{}^{(a \times b)}c} = \Pi_p \{\, \overline{{}^{a}(c+p)} + \overline{{}^{b}(c+\bar{p})} \} \qquad \overline{{}^{a}(b+c)} = \Pi_p \{\, \overline{{}^{a \times p}b} + \overline{{}^{a \times \bar{p}}c} \}$$

$$\overline{(a+b)\circ c} = \Sigma_p \{ \overline{a\circ(c+p)} \times \overline{b\circ(c+\bar{p})} \} \qquad \overline{a\circ(b+c)} = \Sigma_p \{ \overline{(a+p)\circ b} \times \overline{(a+\bar{p})\circ c} \}$$

II. ASSOCIATION FORMULÆ.

1. AFFIRMATIVE.

i. *Simple Formulæ.*

$$\begin{array}{ll}
\overline{{}^{a}(\overline{bc})} = a(bc) = (ab)c = (\overline{\overline{ab}})^{c} & \overline{a(\overline{b^{c}})} = {}^{a}(b^{c}) = ({}^{a}b)^{c} = (\overline{\overline{{}^{a}b}})c \\
\overline{a\circ(\overline{b\circ c})} = a^{(b\circ c)} = {}^{(a\circ b)}c = (\overline{\overline{a\circ b}})\circ c & \overline{a^{(\overline{{}^{b}c})}} = a\circ({}^{b}c) = (a^{b})\circ c = \overline{{}^{(\overline{a^{b}})}c} \\
\overline{a\circ(\overline{bc})} = a^{(bc)} = (a^{b})^{c} = (\overline{\overline{a^{b}}})c & \overline{a(\overline{{}^{b}c})} = {}^{a}({}^{b}c) = {}^{(ab)}c = (\overline{\overline{ab}})\circ c \\
\overline{{}^{a}(\overline{b\circ c})} = a(b\circ c) = ({}^{a}b)\circ c = \overline{{}^{(\overline{{}^{a}b})}c} & \overline{a^{(\overline{b^{c}})}} = a\circ(b^{c}) = (a\circ b)c = (\overline{\overline{a\circ b}})^{c}
\end{array}$$

ii. *Developments.*

(A and E are individual aggregants, and α and ϵ simple components of a and e. The summations and products are relative to all such aggregants and components. The formulæ are of four classes; and for any relative c either all formulæ of Class 1 or all of Class 2, and also either all of Class 3 or all of Class 4 hold good.

CLASS 1.	CLASS 2.
$\overline{a(\overline{bc})} = {}^{a}(bc) = \Pi\{({}^{A}b)c\} = \Pi\{(\overline{\overline{{}^{A}b}})^{c}\}$	$(\overline{\overline{c\circ d}})e = (c\circ d)^{e} = \Pi\{c\circ(d\mathrm{E})\} = \Pi\{\overline{c^{(\overline{d\mathrm{E}})}}\}$
$\overline{a^{(\overline{bc})}} = a\circ(bc) = \Sigma\{(\alpha\circ b)^{c}\} = \Sigma\{(\overline{\overline{\alpha\circ b}})c\}$	${}^{(\overline{\overline{c\circ d}})}e = (c\circ d)\circ e = \Sigma\{c^{({}^{d}\epsilon)}\} = \Sigma\{\overline{c\circ(\overline{{}^{d}\epsilon})}\}$
$\overline{a\circ(\overline{b^{c}})} = a^{(b^{c})} = \Pi\{(a^{b})c\} = \Pi\{(\overline{\overline{a^{b}}})^{c}\}$	$(\overline{\overline{c^{d}}})\circ e = {}^{(c^{d})}e = \Pi\{c\circ(d\circ\epsilon)\} = \Pi\{\overline{c^{(\overline{d\circ\epsilon})}}\}$
$\overline{{}^{a}(\overline{b^{c}})} = a(b^{c}) = \Sigma\{(\mathrm{A}b)^{c}\} = \Sigma\{(\overline{\overline{\mathrm{A}b}})c\}$	$(\overline{\overline{c^{d}}})^{e} = (c^{d})e = \Sigma\{c^{(d\mathrm{E})}\} = \Sigma\{\overline{c\circ(\overline{d^{\mathrm{E}}})}\}$

CLASS 3.	CLASS 4.
$\overline{a^{(\overline{b\circ c})}} = a\circ(b\circ c) = \Sigma\{{}^{(\alpha^{b})}c\} = \Sigma\{(\overline{\overline{\alpha^{b}}})c\}$	${}^{(\overline{\overline{cd}})}e = (cd)\circ e = \Sigma\{{}^{c}(d\circ\epsilon)\} = \Sigma\{\overline{c(\overline{d\circ\epsilon})}\}$
$\overline{a(\overline{b\circ c})} = {}^{a}(b\circ c) = \Pi\{(\mathrm{A}b)\circ c\} = \Pi\{\overline{{}^{(\overline{\mathrm{A}b})}c}\}$	$(\overline{\overline{cd}})e = (cd)^{e} = \Pi\{c(d^{\mathrm{E}})\} = \Pi\{\overline{{}^{c}(\overline{d^{\mathrm{E}}})}\}$
$\overline{{}^{a}(\overline{{}^{b}c})} = a({}^{b}c) = \Sigma\{{}^{(\mathrm{A}b)}c\} = \Sigma\{(\overline{\overline{{}^{A}b}})\circ c\}$	$(\overline{\overline{{}^{c}d}})^{e} = ({}^{c}d)e = \Sigma\{{}^{c}(d\mathrm{E})\} = \Sigma\{\overline{c(\overline{d\mathrm{E}})}\}$
$\overline{a\circ(\overline{{}^{b}c})} = a^{({}^{b}c)} = \Pi\{(\alpha\circ b)\circ c\} = \Pi\{\overline{{}^{(\overline{\alpha\circ b})}c}\}$	$(\overline{\overline{{}^{c}d}})\circ e = {}^{({}^{c}d)}e = \Pi\{c({}^{d}\epsilon)\} = \Pi\{\overline{{}^{c}(\overline{{}^{d}\epsilon})}\}$

The negative formulæ are derived from the affirmative by simply drawing or erasing lines over the whole of each member of every equation.

In order to see the general rules which these formulæ follow, we must imagine the operations symbolized by three marks, as in the commencement of this chapter. We may term the operation uniting the two letters within the parenthesis the *interior* operation, and that which unites the whole parenthesis to the

third letter the *exterior* operation. By *junction-marks* will be meant, in case the parenthesis $\left\{\begin{matrix}\text{follows}\\ \text{precedes}\end{matrix}\right\}$ the third letter, the $\left\{\begin{matrix}\text{first}\\ \text{second}\end{matrix}\right\}$ mark of the symbol of the interior operation and the $\left\{\begin{matrix}\text{second}\\ \text{first}\end{matrix}\right\}$ mark of the symbol of the exterior operation. Using these terms, we may say that the exterior junction-mark and the third mark of the interior operation may always be changed together. When they are the same there is a simple association formula. This formula consists in the possibility of simultaneously interchanging the junction-marks, the third marks, and the exteriority or interiority of the two operations. When the exterior junction-mark and the third mark of the interior operation are unlike, there is a developmental association formula. The general term of this formula is obtained by making the same interchanges as in the simple formulæ, and then changing a to A when after these interchanges ab or ${}^{a}b$ occurs in parenthesis, changing a to α when a^b or $a \circ b$ occurs in parenthesis, changing e to E when de or d^e occurs in parenthesis, and changing e to ϵ when ${}^{d}e$ or $d \circ e$ occurs in parenthesis. When the third mark in the symbol of the exterior operation is affirmative the development is a summation; when this mark is negative there is a continued product.

In the first column of formulæ, the second mark in the sign of the interior operation is a line in Class 1 and a hyphen in Class 3. In the second column, the first mark in the sign of the interior operation is a hyphen in Class 2 and a line in Class 4.

(*To be Continued.*)

NOTE TO PAGE 47.

The relative 0 ought to be considered as at once a concurrent and an alio-relative, and the relative ∞ as at once the negative of a concurrent and the negative of an alio-relative. The statements in the text require to be modified to this extent.

On Certain Ternary Cubic-Form Equations.

By J. J. Sylvester.

CHAPTER I.

Excursus B. — On the Chain Rule of Cubic Rational Derivation.

I think it desirable, while the colors, so to say, are still wet on the palette, and my mind is still dwelling upon the subject which has been casually introduced in the note to the proem contained in the last number of the Journal (and there made use of to determine the number of in-and-exscribed k-laterals to a cubic), without waiting to put forth the titles which in natural order of sequence, perhaps, should immediately follow Title 1 of Section 2, to proceed at once to develop the theory of derivation which, irrespective of the casual use of it alluded to, will be found to be of essential importance when I reach that part of my proposed task which deals with soluble cubic-form equations, nor less so when, in Chapter II., I have to treat of insoluble cases of certain classes of cubic-form equations with four or more terms.

Title 1. — *On the Natural or Discontinuously Numbered Scale of Rational Derivatives to a Point on a Cubic Curve.*

Let us take any point on a cubic curve along with its successive tangentials *ad infinitum.* We may, by drawing straight lines through any two of these points, either contiguous or apart, to meet the curve, obtain an additional set of points, and thus form an enlarged system which may again be subjected to a like process of collineation or tangentialization, and such method of augmentation and amplification may be continued indefinitely. Every point thus obtained will obviously be a rational derivative of the original point (i. e. its co-ordinates will be rational integral functions of those of that point), and, at first sight, it

would seem as if we might in this way obtain a network, or spread,* of rational derivatives; but I shall proceed to show that such is not the case, but that only a line or chain of points will be thus obtained, usually infinite in extent, although for certain positions of the initial point coming to a stop, and in other cases winding round and round upon itself so as still to include only a finite number of distinct points. It will be shown subsequently that, in order to complete the theory of the chain for the purposes of this memoir, it will be necessary to take into account the rational derivatives not merely from a single arbitrary point, but from such points, *combined with a point of inflexion,* and that this additional element will not alter the surprising fact of the absence of reticulation or spread, but merely bring about the insertion into the chain of points corresponding to missing numbers in it as first described, and to the duplication of the chain so completed, owing to every point in it having an opposite point also situated on the curve and collinear with it in respect to the given inflexion. This duplication will be of little importance in general to the arithmetical theory with which we shall be occupied, inasmuch as opposite points will correspond to the same arithmetical values, with merely a change of name between two out of the three variables which denote the co-ordinates of any point. First, let us consider the chain law of derivation when a point on the cubic curve alone is given. I shall call the original point 1, and its first and second tangentials 2 and 4 respectively, and in general use (m, n) to denote the point on a given cubic collinear with two points m, n also situated upon it.† Obviously, then, we shall have $1, 1 = 2 \quad 2, 2 = 4$, using $1, 1 \quad 2, 2$ to denote, in either case, two consecutive points upon the cubic. It is also obvious that if $m, n = p$ then $m, p = n$ and $n, p = m$, so that $1, 2 = 1 \quad 2, 4 = 2$.

Let us call $1, 4 = 5 \quad 2, 5 = 7 \quad 1, 7 = 8 \quad 2, 8 = 10 \quad 1, 10 = 11 \quad 2, 11 = 13$ and so on. It will be seen that no number which is a multiple of 3 is brought into existence by this process. Supposing a, b to be any two integers, neither of them divisible by 3, let us agree to signify by $a \ddagger b$ that of the two values $a + b, a - b$ which is not divisible by 3. The theorem to be established is that the point m, n collinear to m and n will have for its value $m \ddagger n$; as, for instance, $4, 4$, or the third tangential to 1, will have for its value 8, i. e. will be identical with $1, 7$, that is to say, with $1, [2, (1, 4)]$, where 2 and 4 are the first and second

* Spread, as a noun (scarcely to be found in the dictionaries), I employ in the sense in which it occurs in the phrase *spread of foliage.* On this continent the word *spread* is also used to denote a thick coverlet or padded woollen quilt, laid over the bedclothes in winter to keep out the cold; also on both continents as a familiar name for a college banquet.

† Sometimes, however, it will be found more convenient to use $P_1, P_2 \ldots P_n$; $P'_1, P'_2, \ldots P'_n$ in lieu of $1, 2, \ldots n$; $1', 2', \ldots n'$.

tangentials to 1, which amounts to a rule for obtaining the third tangential, when a point on a cubic and its first and second tangentials are given, by collineation alone. The *theory of residuation*, in its simplest form (see Salmon's Higher Plane Curves, 3d ed., p. 134)* teaches us that the rule of the older chemistry known by the name of double decomposition, viz. that $(a,b).(c.d)=(a,c),(b,d)$ is applicable to the same symbols regarded as points on a cubic curve. This rule of double decomposition is all that is required to prove the theorem in question.

Thus, e. g., in order to prove that $1,7=4.4$, I write $1,7=(1,2),(2,5) =(2,2),(1,5)=4,4$. Q. E. D.

So, to prove in general that $r,s=r \ddagger s$ I proceed as follows:

1° Suppose $r=3i+1$; $s=3j+1$, where $j-i$ is positive. Then $r,s=(3i-1,2),(3j+2,1)=(3i-1,1).(3j+2,2)=3i-2,3j+4 =r-3,s+3$. Hence $r,s=r\dot{-}3i,s+3i=1,s+r-1=s+r$.

2° Suppose $r=3i-1$; $s=3j-1$. Then $r,s=(3i-2,1),(3j+1,2) =(3i-2,2),(3j+1,1)=3i-4,3j+2=r-3,s+3$, as before. Hence $r,s=r-3(i-1),s+3(i-1)=2,s+r-2=s+r$.

3° Suppose $r=3i-1$; $s=3j+1$. Then $r,s=(3i-2,1),(3j-1,2) =(3i-2,2),(3j-1,1)=3i-4,3j-2=r-3,s-3$. Hence $r,s= r-3i+3,s-3i+3=2,s-r+2=s-r$.

4° Suppose $r=3i+1$; $s=3j-1$. Then $r,s=(3i-1,2),(3j-2,1) =(3i-1,1),(3j-2,2)=3i-2,3j-4=r-3,s-3$. Hence $r,s= r-3i,s-3i=1,s-r+1=s-r$.

Collecting the four cases, it will be seen that I have proved, for all values of the points r,s in the chain, that $r,s=r \ddagger s$. Q. E. D.

The points 2^j correspond to tangentials of the i^{th} order to the point 1. It is obvious from the above theorem that no process of continued collineation or tangentialization performed upon these points can lead to any points extraneous to the series of points $1,2,4,5,7,8\ldots.$ which form a simple chain extending in general to infinity. Moreover, as it follows from the theory of residuation that any single point reached through the intervention of curves drawn through any number of points on a cubic can be reached by simple linear constructions, it follows that by no conceivable geometrical process can any rational point be reached not included in the numbered chain, and the inference becomes in the

* The theory of residuation was originally brought by me before the Mathematical Society of London, and subsequently, in the form of questions, in the "Educational Times." Dr. Salmon makes no allusion to the fact of my applying the theory to curves of all orders: in the case of the quartic, the residual becomes a system of three points; of a quintic, a system of six points, and so on. I understood Professor H. S. Smith to say that he made use of my theory for the quartic in his memoir which gained half the prize for the subject set by the Academy of Sciences of Berlin, but which I have never seen.

highest degree probable, and, as a matter of fact, is undoubtedly true (although the reasoning upon which it is here made to rest is not absolutely conclusive), that no rational deducts from a *general* point on a *general* cubic exist save those that belong to the numbered chain, the points upon which constitute what may properly be termed a self-contained *group*, infinite or finite (as the case may be) in regard to the number of terms which it contains. I shall presently determine the order of each successive derivative, meaning thereby the order in the co-ordinates of the initial point of any one of the three functions which express the co-ordinates of the derived one.*

The case in which the chain forms a closed polygon, which can only happen when for some number i the i^{th} tangential coincides with the initial point, has already been discussed in the note to the proem.

If the chain is an open but finite one, it is necessary that a tangential of some order shall fall upon a point of inflexion, in which case the succeeding tangentials remain fixed at that point, but otherwise continual new tangentials could be drawn. These are obviously necessary conditions of the chain being finite, whether it be an open chain or winding round upon itself; it remains to show that they are sufficient as well as necessary, but that will best appear after the theory of derivation from a general point combined with a point of inflexion has been discussed.

I shall begin with finding the co-ordinates X, Y, Z of a point on the cubic curve collinear with any two given points x, y, z; ξ, η, ζ. Let $X = \lambda x + \mu\xi$, $Y = \lambda y + \mu\eta$, $Z = \lambda z + \mu\zeta$; then $F(X, Y, Z) = \lambda^3 F(x, y, z) + \lambda^2\mu\left(\xi\frac{d}{dx} + \eta\frac{d}{dy} + \zeta\frac{d}{dz}\right)F(x, y, z) + \mu^3 F(\xi, \eta, \zeta) + \lambda\mu^2\left(x\frac{d}{d\xi} + y\frac{d}{d\eta} + z\frac{d}{d\zeta}\right)F(\xi, \eta, \zeta)$. Hence X, Y, Z will be the collineal to $(x, y, z), (\xi, \eta, \zeta)$ if

$$\lambda : \bar{\mu} :: \left(x\frac{d}{d\xi} + y\frac{d}{d\eta} + z\frac{d}{d\zeta}\right)F(\xi, \eta, \zeta) : \left(\xi\frac{d}{dx} + \eta\frac{d}{dy} + \zeta\frac{d}{dz}\right)F(x, y, z).$$

If now we write $F(x, y, z)$ under its canonical form $x^3 + y^3 + z^3 + Kxyz$, it will be found, on substituting for λ and $\bar{\mu}$ the quantities to which they are proportional, that

* There is a further question, but which, as not material to the object of this memoir, I shall not discuss here, viz. the *degree* in the coefficients of each such derivative. For the tangential, the degórder (being that of the minor determinants of the matrix made up of the differential derivatives of the function and its Hessian) we know to be 4, 4. If x, y, z, be the original co-ordinates, and X, Y, Z, those of the tangential, we know that $F(X, Y, Z)$ being zero when $F(x, y, z)$ (the given cubic) is zero, must be divisible by $F(x, y, z)$. The quotient will be of the degorder 13, 12 — 1, 3, i. e. 12, 9, and is in fact the skew covariant of F.

$$X = (y^2\eta\xi - y\eta^2x + z^2\zeta\xi - z\zeta^2x) + K(yz\xi^2 - \eta\zeta x^2)$$
$$Y = (z^2\zeta\eta - z\zeta^2y + x^2\xi\eta - x\xi^2y) + K(zx\eta^2 - \zeta\xi y^2)$$
$$Z = (x^2\xi\zeta - x\xi^2z + y^2\eta\zeta - y\eta^2z) + K(xy\zeta^2 - \xi\eta z^2).$$

But these expressions admit of a surprising simplification, viz.: we may neglect the terms not containing K, for it will be found that the quantities affected with the coefficient K are to each other in the same ratios as the other three corresponding groups in the values of X, Y, Z. Thus, ex. gr.

$$\begin{aligned}&(yz\xi^2 - \eta\zeta x^2)(z^2\zeta\eta - z\zeta^2y + x^2\xi\eta - x\xi^2y)\\ &- (zx\eta^2 - \zeta\xi y^2)(y^2\eta\xi - y\eta^2x + z^2\zeta\xi - z\zeta^2x)\\ &= (\xi y - x\eta)\{\xi\eta\zeta(x^3 + y^3 + z^3) - xyz(\xi^3 + \eta^3 + \zeta^3)\}\end{aligned}$$

hence $\quad X : Y : Z :: yz\xi^2 - \eta\zeta x^2 : zx\eta^2 - \zeta\xi y^2 : xy\zeta^2 - \xi\eta z^2.$

We might, instead of these simple expressions, take for X, Y, Z the other three groups and (using $x_1y_1z_1$; $x_2y_2z_2$ instead of x, y, z; ξ, η, ζ and (pq) to denote the determinant $p_1q_2 - p_2q_1$) say that X, Y, Z are the minor determinants of

$$\begin{matrix} x_1 \cdot x_2 & y_1 \cdot y_2 & z_1 \cdot z_2 \\ (yz) & (zx) & (xy), \end{matrix}$$

and these are actually the expressions found by Cauchy, and given by him in his *Exercices de Mathématiques*, Paris, 1826, p. 256, l. 18–21, pp. 257–60. I take this reference from a loose page of an article by M. Lucas, but have not access either to that article or to Cauchy's.

It is remarkable that Cauchy should have given quadrinomial expressions for the collineal to two given points on a cubic curve, or their connective, as I shall hereafter term it, when, as shown above, binomial ones fulfil the same purpose. The correctness of these remarkable formulæ admits of easy verification, as follows:

For greater simplicity denote x^3, y^3, z^3, xyz by u, v, w, μ; and $\xi^3, \eta^3, \zeta^3, \xi\eta\zeta$ by u', v', w', μ' respectively. Then $\Sigma(yz\xi^2 - \eta\zeta x^2)^3 = \Sigma(vwu'^2 - v'w'u^2)$ $- 3\mu\mu'\{(u' + v' + w')\mu - (u + v + w)\mu'\} = \Sigma(vwu'^2 - v'w'u^2)$.

Also $\quad K(yz\xi^2 - \eta\zeta x^2)(zx\eta^2 - \zeta\xi y^2)(xy\zeta^2 - \xi\eta z^2)$

$$\begin{aligned}&= -Kxyz(\xi^3\eta^3z^3 + \eta^3\zeta^3x^3 + \zeta^3\xi^3y^3) + K\xi\eta\zeta(x^3y^3\zeta^3 + y^3z^3\xi^3 + z^3x^3\eta^3)\\ &= (u + v + w)(uv'w' + vw'u' + wu'v') - (u' + v' + w')(u'vw + v'wu + w'uv)\\ &= \Sigma(u^2v'w' - u'^2vw).\end{aligned}$$

Hence, giving X, Y, Z the values indicated by the formula, we find

$$X^3 + Y^3 + Z^3 + KXYZ = 0,$$

which equation depends, as seen, and as we know *a priori* must be the case, on the pure algebraical fact that $X^3 + Y^3 + Z^3 + KXYZ$ is a syzygetic function of $x^3 + y^3 + z^3 + Kxyz$ and $\xi^3 + \eta^3 + \zeta^3 + K\xi\eta\zeta$, taking no account of the function $\xi\eta\zeta\,(x^3 + y^3 + z^3) - xyz\,(\xi^3 + \eta^3 + \zeta^3)$, as that is itself a syzygetic function of the two others. If we call the syzygetic multipliers of those two Φ and F respectively, it will at once be seen from what precedes that

$$\Phi = 3\,\xi^2\eta^2\zeta^2 xyz - \xi^3\eta^3 z^3 - \eta^3\zeta^3 x^3 - \zeta^3\xi^3 y^3$$
$$F = 3\,x^3y^3z^3\xi\eta\zeta - x^3y^3\zeta^3 - y^3z^3\xi^3 - z^3x^3\eta^3.\text{*}$$

I now proceed to apply the foregoing results to the problem of determining the order in the co-ordinates of any derivative numbered j (where $j = 3\,i \pm 1$), which may be called its index, and shall prove that *the order of any derivative is the square of its index.*† It will also be shown that each of the derivatives above referred to will be of the form xU, yV, zW, where U, V, W are quantics in x^3, y^3, z^3 as variables, since these quantities satisfy the equation

$$(xU)^3 + (yV)^3 + (zW)^3 + KxyzUVW = 0,$$

where $$Kxyz = -x^3 - y^3 - z^3.$$

From this it follows that, calling x^3, y^3, z^3; a, b, c respectively, the scheme of derivatives contains the various solutions of the algebraico-diophantine equation $aU^3 + bV^3 + cW^3 - (a + b + c)\,UVW = 0$, and that, supposing the law of the squares to be demonstrated, U, V, W will be of the order $\frac{1}{3}\,\{(3\,i \pm 1)^2 - 1\}$, i. e. $3\,i^2 \pm 2\,i$ in a, b, c, where i is any integer. We thus see that the above equation admits of solutions in which U, V, W are of the orders 1, 5, 8, 16, 21, 33, 40 . . . . respectively. It will hereafter be shown, in like manner, that the missing derivatives, whose indices are multiples of 3 (belonging to the arbitrary point and point of inflexion combined), will satisfy the equation

$$U^3 + V^3 + abc\,W^3 - (a + b + c)\,UVW = 0,$$

where U, V, W will be necessarily of the orders $3\,i^2 \pm 2\,i$, $3\,i^2 \pm 2\,i$, $(i \pm 1)\,(3\,i \pm 1)$

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* Thus, $$F = (yz\xi + zx\eta + xy\zeta)\;(yz\xi + \rho zx\eta + \rho^2 xy\zeta)\;(yz\xi + \rho^2 zx\eta + \rho xy\zeta)$$
$$\Phi = (\eta\zeta x + \zeta\xi y + \xi\eta z)\;(\eta\zeta x + \rho\zeta\xi y + \rho^2\xi\eta z)\;(\eta\zeta x + \rho^2\zeta\xi y + \rho\xi\eta z),$$
and it is worthy of notice that we have incidentally solved with quantic values for F, Φ, U, V, W the simultaneous algebraico-diophantine equations

$$U^3 + V^3 + W^3 = (a^3 + b^3 + c^3)\,\Phi - (\alpha^3 + \beta^3 + \gamma^3)\,F$$
$$UVW = abc\Phi - \alpha\beta\gamma\,F.$$

† The proof here supplied is sufficiently exact to dispel any reasonable doubt as to the truth of the law; but an exact proof which does not assume but demonstrates the non-existence of latent common measures to the reduced values of the co-ordinates of the connective to any two derivatives will be furnished under Title 3. — one of the most surprising feats of demonstration which it has ever fallen to the author's lot to accomplish.

respectively, i, as before, representing any integer. Thus we see that, if $a + b + c = 0$, the equations

$$aU^3 + bV^3 + cW^3 = 0 \text{ and } U^3 + V^3 + abcW^3 = 0$$

will admit of an infinite number of solutions in integers, when a, b, c are integer. This fact, as regards the latter equation, has been already pointed out by M. Lucas in this Journal, and previously by the Abbé Pepin in his memoir in Liouville's Journal, 2d series, Tome XV.

Let us begin with applying the formulæ to obtaining the co-ordinates of the tangential.

Let $$x^3 + y^3 + z^3 + 3\,kxyz = 0$$

be the equation to the cubic. If we take $x, y, z;\ x + \delta x, y + \delta y, z + \delta z$ two consecutive points, their connective will be the tangential.

Applying the formulæ just obtained, we shall obtain for its co-ordinates expressions each of the form $P\delta x + Q\delta y + R\delta z$ with only one relation between $\delta x, \delta y, \delta z$. Hence, if we write $\delta z = \lambda\delta x + \mu\delta y$ the resulting ratios must be independent of λ and μ. Consequently we may make $\delta z = 0$. The two connectives then become

$$\begin{gathered} x, y, z \\ x + \delta x, y + \delta y, z, \end{gathered}$$

and the co-ordinates of the tangential will therefore be proportional to

$$yz(x + \delta x)^2 - z(y + \delta y)x^2 : zx(y + \delta y)^2 - z(x + \delta x)y^2 : z^2(xy - (x + \delta x)(y + \delta y))$$

i. e. to $$x(2\,y\delta x - x\delta y) : y(2\,x\delta y - y\delta x) : z(x\delta y + y\delta x)$$

where $$\delta x : \delta y :: y^2 + kxz : x^2 + kyz.$$

Hence the co-ordinates required are as

$$x\{2\,y^3 + x^3 + 3\,kxyz\} : y\{-2\,x^3 - y^3 - 3\,kxyz\} : z(x^3 - y^3),$$

i. e. as $$x\{y^3 - z^3\} : y\{z^3 - x^3\} : z\{x^3 - y^3\},$$

a result which appears to have been first found by Cauchy for the general form, but previously by Euler, and before him by Fermat, for the case $k = 0$.

If we write a, b, c, instead of x, y, z, and call the co-ordinates of the tangential x, y, z, we might find their values by virtue of the condition that the connective of a, b, c and x, y, z is a, b, c over again. This furnishes the equations

$$\begin{aligned} bcx^2 - a^2yz &= am \\ cay^2 - b^2zx &= bm \\ abz^2 - c^2xy &= cm, \end{aligned}$$

which may be satisfied by writing

$$x = a(b^3 - c^3)\rho \;;\; y = b(c^3 - a^3)\rho \;;\; z = c(a^3 - b^3)\rho\,;$$
$$(a^6 + b^6 + c^6 - a^3b^3 - b^3c^3 - a^3c^3)\,\rho^2 = m\,;$$

but whether or not the above is necessarily the only possible solution is not quite clear *a priori*, and *a posteriori* it looks as if the solutions might be manifold.

The co-ordinates of the point whose index is 4, i. e. of the second tangential, will be those of the first tangential to the point

$$x(y^3 - z^3) \;:\; y(z^3 - x^3) \;:\; z(x^3 - y^3), \text{ viz.}$$
$$x(y^3 - z^3)\{y^3(x^3 - z^3)^3 + z^3(x^3 - y^3)^3\} \;:\; y(z^3 - x^3)\{z^3(y^3 - x^3)^3 + x^3(y^3 - z^3)^3\}$$
$$:\; z(x^3 - y^3)\{x^3(z^3 - y^3)^3 + y^3(z^3 - x^3)^3\},$$

and are of the order 16.

To find the co-ordinates of the point whose index is 5, we may take the connective of the one last found, and of x, y, z, i. e. of 4 and 1. Let us call them xU, yV, zW, and, for greater simplicity, denote x^3, y^3, z^3, for u, v, w. Then, omitting the common factor xyz,

$$U = (v - w)^2\{v(u - w)^3 + w(u - v)^3\}^2$$
$$-(w - u)\;(u - v)\{w(v - u)^3 + u(v - w)^3\}\;\{u(w - v)^3 + v(w - u)^3\},$$

with similar quantities (*mut. mut.*) set against V and W.

These quantities will have the common measure $u^2 + v^2 + w^2 - uv - uw - vw$.

To prove this let either one of its factors as $u + \rho v + \rho^2 w = 0$.

Then $\qquad v - u = \rho^2(w - v)$ and $u - w = \rho(w - v)$,

and the representative of U above written becomes

$$\{(v - w)^2 - (w - u)\,(u - v)\}(w - v)^8 = (v^2 + w^2 + u^2 - vw - uw - uv)\,(w - v)^8 = 0.$$

Hence the representative of U vanishes with, and therefore contains

$$u^2 + v^2 + w^2 - uv - uw - vw$$

as a factor, and the same must evidently be true for the representatives of V and W; hence, U, V, W, will be of the order $10 - 2$ or 8, in u, v, w, and the co-ordinates xU, yV, zW, of the order $3 . 8 + 1$, i. e. of the order 25 in xyz.

The preceding demonstration depends essentially on the fact that my simplified formulæ for the co-ordinates of the connective of two points on a cubic fail, that is to say, become illusory, for a particular relation between the two points, as is easily seen; for let x, y, z; $x, \rho y, \rho^2 z$ be two points on a cubic, then the formulæ for X, Y, Z, the connective's co-ordinates, become

$$(\rho y \,.\, \rho^2 z - yz)\,x^2; \quad (\rho^2 z \,.\, x - xz\rho^2)\,y^2; \quad (x \,.\, \rho y - xy\rho^4)\,z^2.$$

i. e. all vanish, whereas it may be remarked that the general expressions given at page 62,

$$X = (y^2\eta\xi - y\eta^2x + z^2\zeta\xi - z\zeta^2x) + K(yz\xi^2 - \eta\zeta x^2)$$
$$Y = (z^2\zeta\eta - z\zeta^2y + x^2\xi\eta - x\xi^2y) + K(zx\eta^2 - \zeta\xi y^2)$$
$$Z = (x^2\xi\zeta - x\xi^2z + y^2\eta\zeta - y\eta^2z) + K(xy\zeta^2 - \xi\eta z^2),$$

become the minors of

$$\begin{matrix} x^2 & \rho y^2 & \rho^2z^2 \\ (\rho^2-\rho)\,yz & (1-\rho^2)\,zx & (\rho-1)\,xy\,; \end{matrix}$$

i. e. $(\rho^2-\rho)\,x\,(y^3-z^3)$, $(\rho-1)\,y\,(z^3-x^3)$, $(1-\rho^2)\,z\,(x^3-y^3)$, which are the same as

$$x\,(y^3-z^3),\quad \rho^2y\,(z^3-x^3),\quad \rho z\,(x^3-y^3),$$

and remain perfectly valid.

This law of the failing case enables me to prove very easily the *Law of Squares*, as follows:

Suppose it proved that for all indices inferior to $6\,i$ the order of the derivative is equal to the square of its index; then, to prove that the same law is true up to $6\,(i+1)$, it is only necessary to consider the cases of $6\,i+1$, $6\,i+5$, for, as regards the index $6\,i+2$ and $6\,i+4$, the derivatives may be regarded as the tangentials of the derivatives to indices $3\,i+1$ and $3\,i+2$, and will consequently be of the orders $4\,(3\,i+1)^2$ and $4\,(3\,i+2)^2$, i. e. $(6\,i+2)^2$ and $(6\,i+4)^2$ respectively.

Let us further suppose that for derivatives whose indices are inferior to $6\,i$ the co-ordinates are of the form xU, yV, zW; U, V, W being quantics in x^3, y^3, z^3; then, obviously, from the mode of forming the tangential, this will be true for derivatives whose indices are $6\,i+2$, $6\,i+4$: for the tangential to xU, yV, zW is $xU\,(y^3V^3 - z^3W^3),\, yV\,(z^3W^3 - x^3U^3),\, zW\,(x^3U^3 - y^3V^3)$.

Let us consider the point (1) whose co-ordinates x, y, z satisfy the equation

$$x^3 + \rho y^3 + \rho^2z^3 = 0.$$

For such a point $\qquad y^3 - z^3 : z^3 - x^3 : x^3 - y^3 :: 1 : \rho : \rho^2,$

and the point (2) becomes $x, \rho y, \rho^2z$. Consequently the point (4) becomes $x\,(y^3-z^3), \rho y\,(z^3-x^3), \rho^2z\,(x^3-y^3)$ the same as $x, \rho^2y, \rho z$; hence the point (5), the connective of (1, 4), becomes $x\,(y^3-z^3), \rho y\,(z^3-x^3), \rho^2z\,(x^3-y^3)$, the same as $x, \rho^2y, \rho z$, so that, denoting the derivatives by their indices,

$$5 = 4 \qquad 7 = 1, 8 = 1, 1 = 2 \qquad 10 = 2, 8 = 2, 1 = 1$$
$$11 = 4, 7 = 4, 2 = 2 \qquad 13 = 2, 11 = 2, 2 = 4, \text{ etc.}$$

We have, thus, for all values of the point i

$$9i \pm 1, 2, \pm 4 = 1, 2, 4,$$

when 1 is the point for which $x^3 + \rho y^3 + \rho^2 z^3 = 0$.

Hence, if p, p' be any two points for which $p - p' = \dot{3}$, then p, p' will be respectively identical with some two out of the three points $1, 2, 4$. And it will at once be seen that the simplified formulæ for the connective of any two of these three points become illusory.

Now the point $6i + 1$ is the connective of $3i - 1$ and $3i + 2$, and the point $6i + 5$ is the connective of $3i + 1$ and $3i + 4$.

Hence, in each of these cases, the simplified formulæ become illusory, i. e. the expressions for each of the co-ordinates vanish when $x^6 + y^6 + z^6 - x^3y^3 - x^3z^3 - y^3z^3$ vanishes, and must therefore contain it as a common measure. Moreover, the simplified formulæ for the connective co-ordinates for the points xU, yV, zW; xU', yV', zW' will contain x^2yz, y^2zx, z^2xy, and will therefore have the common measure xyz. Hence the values of the co-ordinates when freed from these common measures will be of the order in x, y, z, $2(3i - 1)^2 + 2(3i + 2)^2 - 9$ for the point $6i + 1$, and $2(3i + 1)^2 + 2(3i + 4)^2 - 9$ for the point $6i + 5$, i. e. $(6i + 1)^2$ and $(6i + 5)^2$ respectively, and will obviously continue to be quantics in x^3, y^3, z^3 multiplied by x, y, z respectively. Hence the theorem being true for index inferior to 6 is true universally.

It will be observed that any co-ordinate X of the point k must contain the X co-ordinate of the point k' where k' is any factor of k; for if $k = \delta k'$ the point k may be obtained by forming the point δ to the point k', and it has been shown that the δ derivative to any point has co-ordinates which contain respectively those of the initial point. Consequently the X co-ordinate to any point k may be resolved into factors containing a primitive part of the order τk (the totient of k) in the variables, and a non-primitive part containing the primitive part of each power of a prime contained in k, and with the exception of the single factor x all the others will be quantics in x^3, y^3, z^3; and, of course, the same remark applies to the other two co-ordinates Y and Z. We might obtain the point $m \ddagger n$ as the connective of m, n. In that case the simplified formulæ would give expressions of the order $2(m^2 + n^2)$ in x, y, z; and as the actual order of the co-ordinates in those variables is $(m \ddagger n)^2$, it follows that when $m - n \equiv 0$, Mod. 3, there will be a common measure (a symmetrical function of x, y, z) of the order $(m - n)^2$, and when $m + n \equiv 0$, Mod. 3, of the order $(m + n)^2$ running through those expressions, and it might be desirable to ascertain its form; but without waiting to solve

this problem,* which is irrelevant to the matter in hand, I shall proceed at once to consider the derivatives corresponding to indices which are multiples of the number 3, to obtain which it is only necessary, as will be seen immediately, to combine one given point of inflexion with one arbitrary point of the curve. But, before doing so, it may be well to notice, that while the preceding investigation serves to show that the abridged formulæ for the connective co-ordinates possess the common measure $xyz\,(x^6 + y^6 + z^6 - x^3y^3 - x^3z^3 - y^3z^3)$, it does not demonstrate categorically that there is no other; or that some power of the second factor above written other than the first might not be a common measure. Consequently, what we have strictly proved, as will be evident on reviewing the argument, is that the order to a derivative of the index $3i \pm 1$ cannot *exceed* the square of that index; but before I come to an end of the discussion I trust to be able to establish with *Dirichletian* rigor that the order is actually equal to the square of the index.†

Title 2. — *On the Completed or Continuously Numbered Scale of Rational Derivatives to an Arbitrary Point on a Cubic, of which one Point of Inflexion is given.*

Let I be the given point of inflexion, and let any point (or system of points) and another point (or system of points respectively) collinear with the former in respect to I be called opposites. It is obvious that $I, I = I$, or that the inflexion is its own opposite. It will be convenient to denote the opposite to any point by the same index, but accented.

We have, then, obviously,

$$p', p = I;\ (p')' = p \text{ and } (p', q)' = I\,(p', q) = (I, I), (p', q) = (I, p'), (I, q) = p, q'.$$

Let $I', 2 = 3$; $I', 5 = 6$; and in general $I', 3i - 1 = 3i$. This is matter of definition. Let, now, the infinite system $1, 2, 3, 4, 5, 6, 7 \ldots$ and its opposite be regarded as a single group. I say, 1°, that this will be a closed group, in the sense that a straight line drawn through any two points (contiguous or apart) of this double chain will cut the cubic in a third point included in the group.

2° That the new points will be rational in respect to the co-ordinates of the initial point and the given point of inflexion, and, 3°, that the order in the variables for every point, without regard to its relation to the modulus 3, will be, as before, the square of its index.

* It is completely solved in the corollary to Title 5.

† This anticipation (for it was only such when these words were written) will be found fully realized under Title 5.

I proceed to show that the connective of any two points in the double chain may be expressed as a single point therein. Several cases present themselves according to the form of each of the two connected points in respect to the modulus 3 except when the indices are congruent in respect to that modulus.

When the residues (r, r'), in respect to that modulus, are dissimilar, the result will in general be different according as one of them (as r) belongs to the higher or lower index.

In what follows it is to be understood that $i \gtreqless j$.

Theorem 1. To prove that $3i+1, (3j+1)' = 3j - 3i$

and $$3i+2, (3j+2)' = (3j-3i)'.$$

[This will imply that $(3i+1)', 3j+1 = (3j-3i)'$ and $(3i+2)', 3j+2 = 3j-3i$].

$$3i+1, (3j+1)' = (3i-1, 2), [(3j-1)', 2'] = (2, 2'), [3i-1, (3j-1)']$$
$$= (3i-1)', 3j-1 = [(3i-2)', 1'], (3j-2, 1) = (1, 1'), [(3i-2)', 3j-2]$$
$$= 3i-2, (3j-2)'.$$

Hence, $3i+1, (3j+1)' = 1, (3j-3i+1)' = (1, 2), [(3j-3i-1)', 2']$
$$= (2, 2'), [1, (3j-3i-1)'] = 1', (3j-3i-1) = 3j-3i$$

and $$3i-1, (3j-1)' = I, [3i-2, (3j-2)'] = (3j-3i)'.$$

Theorem 2. To prove that $3i+1, (3j-1)' = (3i+3j)'$

and $$3i-1, (3j+1)' = 3i+3j.$$

[This will imply that $(3i+1)', 3j-1 = 3i+3j$

and $$(3i-1)', 3j+1 = (3i+3j)'].$$

$$3i+1, (3j-1)' = 3i-1, 2;\ (3j+1)', 2' = (3i-1)', 3j+1$$
$$= [(3i-2)', 1'] (3j+2, 1) = 3i-2, (3j+2)'.$$

Therefore, $3i+1, (3j-1)' = 1, (3j+3i-1)' = (3i+3j)'$

and $$3i-1, (3j+1)' = I, [(3i-1)', 3j+1] = 3i+3j.$$

Collecting the results of these two theorems, we see that

$$\left.\begin{array}{ll} & 3i \pm 1, (3j+1)' = 3j \mp 3i = (3i \mp 1)', 3j-1 \\ \text{and} & 3i \pm 1, (3j-1)' = (3j \pm 3i)' = (3i \mp 1)', 3j+1 \end{array}\right\}, \qquad \text{(A)}$$

so that, using $p \overset{\circ}{-} q$ (where neither p nor q contains 3), to denote that one of the two numbers $p+q, p \sim q$, which is divisible by $3, (p, q')$ is always either $p \overset{\circ}{-} q$ or $(p \overset{\circ}{-} q)'$. Also

$$3i+1, (3j)' = (3i-1, 2), [1, (3j-1)'] = (1, 2), [3i-1, (3j-1)']$$
$$= (3j-3i)', 1 = (1', 3j-3i+1), (2, 1) = [(1', 1), (3j-3i+1, 2)]$$
$$= (3j-3i-1)';$$

again $\quad 3\,i\,,(3j+1)' = (3\,i-1\,,1')\,,[(3j-1)',2'] = 1',(3j-3\,i)'$
$= (1',2')\,,[1\,,(3j-3\,i-1)'] = (1\,,1')\,,[(3j-3\,i-1)',2'] = 3j+1-3\,i;$

and lastly $\quad 3\,i\,,(3\,i+1)' = (3\,i-1\,,1')\,,[(3\,i-1)',2'] = I\,,1' = 1.$

Hence, collecting the results, $3\,\iota\,,(3\,i+1)' = (3\,i+1) \sim 3\,\iota$, whatever the relation of magnitude may be between i and ι.
Similarly,

$$3\,i-1\,,(3j)' = (3\,i+1\,,2)\,,[1\,,(3j-1)'] = (1\,,2)\,,[3\,i+1\,,(3j-1)'] = 1\,,(3\,i+3j)' = (3\,i+3j-1)';$$

$$(3\,i)',3j-1 = [(3\,i-1)',1]\,,(3j+1\,,2) = 1\,,(3\,i+3j)' = (3\,i+3j-1)';$$

and $\quad (3\,\iota)',3\,i-1 = [(3\,i-1)',1]\,,(3\,i+1\,,2) = 1\,,(6\,i)' = (6\,i-1)'.$

Hence, collecting the results, $3\,i-1\,,(3\,\iota)' = (3\,i+3\,\iota-1)'$, and we have

$$\left.\begin{array}{l} 3\,\iota\,,(3\,i+1)' = (3\,i+1) \sim 3\,\iota\,;\,(3\,\iota)',3\,i+1 = [(3\,i+1) \sim 3\,\iota]'; \\ 3\,\iota\,,(3\,i-1)' = 3\,i-1+3\,\iota\,;\,(3\,\iota)',3\,i-1 = (3\,i-1+3\,\iota)'. \end{array}\right\}\ (\mathrm{B})$$

Also,

$$\left.\begin{array}{l} 3\,\iota\,,3\,i-1 = (3\,\iota-1\,,1')\,,(3\,i-2\,,1) = (3\,\iota-1\,,3\,i-2)' = [(3\,i-1) \sim 3\,\iota]' \\ 3\,\iota\,,3\,i+1 = (3\,\iota-1\,,1')\,,(3\,i+2\,,1) = (3\,\iota-1\,,3\,i+2)' = (3\,\iota+3\,i+1)'. \end{array}\right\}\ (\mathrm{B}')$$

It remains only to determine the connectives of $3\,i\,,3\,\iota$ and of $3\,i\,,(3j)'$ or $(3\,i)',3j$, which is easily done, for

$$3\,i\,,3\,\iota = (3\,i-1\,,1')\,,(3\,\iota-1\,,1') = (1',1')\,,(3\,i-1\,,3\,\iota-1) = 2',3\,i+3\,\iota-2.$$

Hence (by A) $3\,i\,,3j = (3\,i+3j)'$ and consequently $(3\,i)',(3\,\iota)' = 3\,i+3\,\iota$.

Again $3\,i\,,(3j)' = (3\,i-1\,,1')\,,[(3j-1)',1] = (1\,,1')\,,[3\,i-1\,,(3j-1)'] =$ (by theorem A) $I\,,(3j-3\,i)' = 3j-3\,i$. Hence also $3j\,,(3\,i)' = (3j-3\,i)'$.

These three results may be designated theorem C, and theorems $A\,,B\,,B',C$ collectively prove that the original scale $1\,,2\,,4\,,5\,,7\,,8\ldots.$, which formed a closed system (so to say "group"), remains still closed when we complete it by insertion of multiples of 3, provided that we join on to the completed system $1\,,2\,,3\,,4\,,5\,,6\,,7\ldots.$ the opposite system $1',2',3',4',5',6',7'\ldots.$

In every case it will be observed the connective of two indices (disregarding the accent) is either their sum or their difference.

The double scale may be formed by alternate addition of 1 and $1'$ in the manner following:

$$1\,,1=2 \qquad 1',2=3 \qquad 1\,,3=4' \qquad 1',4'=5' \qquad 1\,,5'=6' \qquad 1',6'=7$$
$$1\,,7=8 \qquad 1',8=9 \qquad 1\,,9=10' \qquad 1',10'=11' \qquad 1\,,11'=12'$$

.... which gives the numbers $1, 2, 3, 4', 5', 6', 7, 8, 9, 10', 11', 12'$, etc.; and, in like manner, by interchanging $1, 1'$, we may obtain $1', 2', 3', 4, 5, 6, 7', 8', 9', 10, 11, 12$, etc.

The new points $3, 6, 9 \ldots\ldots; 3', 6', 9' \ldots\ldots$ belong to the natural scales $1, 2, 5 \ldots\ldots; 1', 2', 5' \ldots\ldots$ collectively and not respectively; and the accented and unaccented multiples of 3 might have had their significations interchanged without any impropriety. It is now necessary to extend the law of the order in the variables to these inserted points, and to prove that for them, as for the points in the natural scale, the order of any point, in the variables of the initial point, is the square of its index.

If the cubic be thrown into the canonical form $x^3 + y^3 + z^3 + kxyz$, the point $x = 1, y = -1, z = 0$ may be taken to represent I, and if x, y, z be the initial point 1, the co-ordinates of $1'$ (the connective of 1 and I) become by the general formula yz, zx, z^2, or, more simply, y, x, z.

To find 3, then, we have to take the connective of y, x, z and $x(y^3 - z^3)$, $y(z^3 - x^3), z(x^3 - y^3)$; its co-ordinates, accordingly, by the general formula, are

$$\begin{gathered} yz\,(z^3 - x^3)(x^3 - y^3)\,y^2 - x^2z\,(y^3 - z^3)^2 \\ xz\,(x^3 - y^3)(y^3 - z^3)\,x^2 - y^2z\,(z^3 - x^3)^2 \\ xy\,(y^3 - z^3)(z^3 - x^3)\,z^2 - yxz^2\,(x^3 - y^3)^2; \end{gathered}$$

or, neglecting the common factor z, the co-ordinates of 3 are

$$\begin{gathered} y^3\,(x^3 - y^3)(x^3 - z^3) + x^3\,(y^3 - z^3)^2 \\ x^3\,(y^3 - x^3)(y^3 - z^3) + y^3\,(z^3 - x^3)^2 \end{gathered}$$

and $$xyz\,(z^3 - x^3)(z^3 - y^3) + xyz\,(x^3 - y^3)^2;$$

or $$\begin{gathered} y^3x^6 + z^3y^6 + x^3z^6 - 3\,x^3y^3z^3 \\ x^3y^6 + z^3x^6 + y^3z^6 - 3\,x^3y^3z^3 \end{gathered}$$

and $$xyz\,(z^6 + y^6 + x^6 - x^3y^3 - z^3x^3 - y^3z^3).$$

In the particular case where $x^3 + y^3 + z^3 = 0$, these expressions (writing for greater brevity L, M, N for x^3, y^3, z^3) become

$$\begin{gathered} ML^2 - (L + M)\,M^2 + L\,(L + M)^2 + 3\,LM\,(L + M) \\ LM^2 - (L + M)\,L^2 + M\,(L + M)^2 + 3\,LM\,(L + M) \\ xyz\,[(L + M)^2 + L^2 + M^2 - LM + (L + M)^2] \end{gathered}$$

or $$\begin{gathered} L^3 + 6\,L^2M + 3\,LM^2 - M^3 \\ M^3 + 6\,M^2L + 3\,ML^2 - L^3 \\ 3\,xyz\,(L^2 + LM + M^2); \end{gathered}$$

which remain equally good, as co-ordinates of the point 3 to the initial point x, y, z, when the cubic is $x^3 + y^3 + Cz^3$, as is easily seen by writing $C^{\frac{1}{3}}z = \zeta$.

The point 3, it follows from what precedes, is of the order 9 in the variables x, y, z, and the same will be true for $3'$, which is obtained from 3 by the interchange of x and y; but in order that these points may be arithmetically as well as algebraically rational, it is of course necessary that the given cubic may admit of being expressed under the form $Ax^3 + Ay^3 + Cz^3 + Kxyz$, where A, C and K are integers.

Again, since $6 = 3', 3'$, 6 is the 2 of $3'$, and similarly $6'$ is the 2 of 3; since $9 = 3', 6'$ and $6'$ is the 2 of 3, 9 is the 3 of 3. So again, since $12 = 3', 9'$ and $9'$ is the 3 of $3'$, 12 is the $(1, 3)$ of $3'$, i. e. the $4'$ of $3'$ or 4 of 3; and similarly $12'$ is the 4 of $3'$. So again,

$$15 = (3', 12') = (1, 4) \text{ of } 3' = 5 \text{ of } 3', \text{ and } 15' = 5 \text{ of } 3$$
$$18 = (3', 15') = (1, 5') \text{ of } 3' = 6' \text{ of } 3' = 6 \text{ of } 3, \text{ and } 18' = 6 \text{ of } 3'$$
$$21 = (3', 18') = (1, 6) \text{ of } 3' = 7' \text{ of } 3' = 7 \text{ of } 3, \text{ and } 21' = 7 \text{ of } 3'$$
$$24 = (3', 21') = (1, 7) \text{ of } 3' = 8 \text{ of } 3', \text{ and } 24' = 8 \text{ of } 3;$$
$$27 = (3', 24') = (1, 8') \text{ of } 3' = 9' \text{ of } 3' = 9 \text{ of } 3 \dots.$$

Hence, in general,

$$9i + 3 = (3i + 1) \text{ of } 3; \quad 9i + 6 = (3i + 2)' \text{ of } 3; \text{ and } 9i = 3i \text{ of } 3.$$

Consequently

$$3^q(3i + 1) = (3i + 1) \text{ of } 3 \text{ of } 3 \text{ of } 3 \dots. (q \text{ times repeated}),$$
and
$$3^q(3i + 2) = (3i + 2)' \text{ of } 3 \text{ of } 3 \text{ of } 3 \dots. (q \text{ times repeated}).$$

From this it follows, obviously, that $3^q(3i \pm 1)$ and $[3^q(3i \pm 1)]'$ are each of the order $[3^q(3i \pm 1)]^2$ in the variables, and thus the law of the squares extends to all points alike in the completed scale.

TITLE 3. — *On Compound Derivation.*

The object of what follows is to show that any derivative of a derivative has for its index (due regard being paid to the accents) the product of the numerical values of the indices of the operator and operand derivatives, that is to say, the $i^?$ of $j^? = ij^?$; the mark of interrogation denoting either a blank or an accent, as the case may be. Thus, while connection involves addition or subtraction, composition involves a process of multiplication.

1° Let us consider the i of j when neither i nor j contains 3. Then

$3k + 1$ of $j = (2$ of $j), (3k - 1$ of $j)$ and $3k + 2$ of $j = (1$ of $j), (3k + 1$ of $j)$.

Suppose the theorem proved up to $3k-1$. Then

$$3k+1 \text{ of } j = 2j, 3kj - j = (3k+1)j$$
$$3k+2 \text{ of } j = j, 3kj + j = (3k+2)j.$$

Hence it is true up to $3(k+1)-1$, and, being true when $k=1$ (since 1 of $j=j$ and 2 of $j = j, j = 2j$), it is true universally.

In like manner, since 1 of $j' = j'$ and 2 of $j' = j', j' = I, (j, j) = (2j)'$, it may be shown that i of $j' = (ij)'$. Moreover

$$1' \text{ of } j = j', \text{ and therefore } 2' \text{ of } j = (1' \text{ of } j), (1' \text{ of } j) = j', j' = 2j'$$

and $$(3k+1)' \text{ of } j = (2' \text{ of } j), [(3k-1)' \text{ of } j]$$
$$(3k+2)' \text{ of } j = (1' \text{ of } j), [(3k+1)' \text{ of } j];$$

so that, if the equation i' of $j = (ij)'$ holds good up to $i = 3k-1$,

$$(3k+1)' \text{ of } j = [(3k+1)j]', \text{ and } (3k+2)' \text{ of } j = [(3k+2)j]';$$

so that the equation i' of $j = (ij)'$ will hold good up to $3(k+1)-1$, and, being true for $k=1$, is true universally.

In like manner, since $1'$ of $j' = j$, it will follow that i' of $j' = ij$.

It remains to obtain the corresponding equations when i, j are one or both of them multiples of 3.

Since 3 of $3^q = (3^q, 3^q), (3^q)' = (2 \cdot 3^q)', (3^q)' = 3^{q+1}$,

$$9 \text{ of } 3^q = 3 \text{ of } 3 \text{ of } 3^q = 3 \text{ of } 3^{q+1} = 3^{q+2}$$
$$27 \text{ of } 3^q = 3 \text{ of } 9 \text{ of } 3^q = 3 \text{ of } 3^{q+2} = 3^{q+3}, \text{ and so on.}$$

Hence $$3^p \text{ of } 3^q = 3^{p+q}.$$

Again, $$3 \text{ of } 3j+1 = (3j+1, 3j+1), (3j+1)'$$
$$= 6j+2, (3j+1)' = 9j+3 \text{ by A.}$$

Hence 3^2 of $3j+1 = 3$ of $9j+3 = (18j+6)', (9j+3)' = 27j+9$ by C,

3^3 of $3j+1 = 3$ of $27j+9 = (54j+18)', (27j+9)' = 81j+27$ by C,

and so on. Hence $$3^p \text{ of } 3j+1 = 3^p(3j+1).$$

Again, $$3 \text{ of } 3j+2 = (3j+2, 3j+2), (3j+2)'$$
$$= 6j+4, (3j+2)' = (9j+6)' \text{ by A.}$$

Hence 3^2 of $3j+2 = 3$ of $(9j+6)' = 18j+12, 9j+6 = (27j+18)'$ by C,

and so on. Hence $$3^p \text{ of } 3j+2 = [3^p(3j+2)]'.$$

Again, $$3j+1 \text{ of } 3^p = (2 \text{ of } 3^p), (3j-1 \text{ of } 3^p) = (3^p, 3^p), (3j-1 \text{ of } 3^p)$$
$$= (2 \cdot 3^p)', (3j-1 \text{ of } 3^p)$$

and $$3j-1 \text{ of } 3^p = (1 \text{ of } 3^p), (3j-2 \text{ of } 3^p).$$

Suppose it true that $3j-2$ of $3^p = (3j-2)\,3^p$ for a certain value of j.

Then $\quad 3j-1$ of $3^p = 3^p, (3j-2)\,3^p = [(3j-1)\,3^p]'$

and $\quad 3j+1$ of $3^p = (2\,.\,3^p)', [(3j-1)\,3^p]' = (3j+1)\,3^p$.

But 1 of $3^p = 1\,.\,3^p$; hence, for all values of j,

$$3j+1 \text{ of } 3^p = (3j+1)\,3^p = 3^p \text{ of } 3j+1$$
$$3j-1 \text{ of } 3^p = [(3j-1)\,3^p]' = 3^p \text{ of } 3j-1.$$

Hence, by the well-known method of successive transformation, we obtain the following results:

When neither m nor n contains 3, when both contain 3, and when one of them contains 3 and the other is of the form $3j+1$, we have

$$m \text{ of } n = n \text{ of } m = m' \text{ of } n' = n' \text{ of } m' = mn$$
$$m \text{ of } n' = n' \text{ of } m = m' \text{ of } n = n \text{ of } m' = (mn)'.$$

In the remaining case (viz. when of m and n, one contains 3 and the other is of the form $3j-1$), we have

$$m \text{ of } n = n \text{ of } m = m' \text{ of } n' = n' \text{ of } m' = (mn)'$$
$$m \text{ of } n' = n' \text{ of } m = m' \text{ of } n = n \text{ of } m' = mn.$$

This completes the algorithm of rational derivation.

TITLE IV.— *On Pertactile or Periodic Points on a Cubic Curve.*

A pertactile point, or point of pluperfect tactility, on a general cubic is a point at which the cubic admits of a higher order of contact with another curve than is in general possible. Thus the points of inflexion are pertactile points, because a tangent at one of them will meet the curve in three consecutive points. The same is the case with Plücker's twenty-seven points, because at each of them a conic of closest contact will pass through six consecutive points, the sixth point in which any conic passed through five consecutive points cuts the curve coinciding, in this case, with the point of contact. So, in general, a curve of the i^{th} order can only be made to pass through $3i-1$ consecutive points situated at P; but if the i^{th} derivative of P is a point of inflexion, then the $3i^{\text{th}}$ point common to all curves of the i^{th} order passing through $3i-1$ consecutive points at P will coincide with P, so that such curves will pass through $3i$ consecutive points, and P may accordingly be termed a point of pluperfect tactility, or more briefly, a pertactile point.

To prove that this is the case, it is necessary, in the first place, to prove that, at a general point P in the cubic, the $3i^{\text{th}}$ point in which all curves of the i^{th} order passing through $3i-1$ consecutive points at P intersect the cubic, is the $(3i-1)^{\text{th}}$ derivative of P, which may be done inductively as follows:

Suppose P_{3i-1} is the residual of $3i-1$ consecutive points at P. To find the residual of $3i+2$ consecutive points there, we may combine $3i-1$ giving the residual P_{3i-1}, two more of them giving the residual P_2, and one giving Q, R, any two points collinear with P. We then combine $(P_{3i-1}, P_2), (Q, R)$ and obtain P_{3i+1}, P_1 which gives P_{3i+2} as the required residual. Hence the theorem, being true for P_2 (the residual of two consecutive points at P) and true for $P_{3(i+1)-1}$ if true for P_{3i-1}, is true universally.

If, now, the residual of $3i-1$ points at P is to fall at P we must have $P_1 = P_{3i-1}$.

1° Suppose $i = 3k-1$, then $P_1, P_{i-1} = P_{i-1}, P_{3i-1}$, i. e. $P_i = P_{2i}$.

Hence P_i is a point of inflexion I, or, as we may express it, P is an i^{th} sub-derivative of such point, or $P = I_{\frac{1}{i}}$.

2° Suppose $i = 3k+1$, then $P_1, P_2 = P_2, P_{3i-1}$, i. e. $P_1 = P_{3i+1}$.

Hence $P_1, P_{i+1} = P_{i+1}, P_{3i+1}$, i. e. $P_i = P_{2i}$, and, as before, $P = I_{\frac{1}{i}}$.

3° Suppose $i = 3k$.

Then $1, (i-1)' = (i-1)', 3i-1$, i. e. $i' = 2i = i', i'$. Consequently i', and therefore also i, is a point of inflexion.

Hence, as in the other two cases, P is an i^{th} sub-derivative of a point of inflexion,* which may either be the point used to form the scale, or any of the eight other inflexions.†

It may be well to notice here that whilst P_i, when i does not contain 3, is, as already shown, of the form xU, yV, zW, it follows from the law of compound derivation, since P_3 is of the form $R, S, xyz\Theta$ (where R, S, Θ, like U, V, W, are quantics in x^3, y^3, z^3) that P_i, when i is a multiple of 3 or any power of 3, will be of the form $M, N, xyz\Omega$ (where M, N, Ω are still quantics in x^3, y^3, z^3).

Calling X, Y, Z any i^{th} derivative to $x^3 + y^3 + z^3 + kxyz = 0$, we must have $X^3 + Y^3 + Z^3 + kXYZ = 0$; and, in order for such derivative to be a point of inflexion, it is necessary and sufficient that $X = 0$ or $Y = 0$ or $Z = 0$; combining these equations respectively with the given cubic, we shall obtain, in all, 3 times $3i^2$ or $9i^2$ points, sub-derivatives of the i^{th} grade to one or other of the inflexions; but out of these, whether i be or be not divisible by 3, nine will correspond to $x = 0$, $y = 0$, or $z = 0$ combined with the curve, i. e. will be the points of inflexion themselves. Moreover, unless i be a prime number, it follows from the law of compound derivation, combined with the fact that x, y, z enter distributively or collectively into the derived co-ordinates X, Y, Z, that, if i' be any

* A sub-derivative of an inflexion may conveniently be termed a sub-inflexion.

† The above formulæ show that $i, i' = 3i = 3i'$; hence $3i$ and $3i'$ coincide with the original point of inflexion, whereas $i, i', 2i, 2i'$ need not coincide with the original point of inflexion.

factor of i, and X', Y', Z' the co-ordinates of the i'^{th} derivative, Z will contain Z' and X, Y or Y, X, will contain X', Y' respectively. There will thus be a *primitive* part to X, Y, Z which results from driving out all the factors corresponding to any factor of i (unity included), and, if we suppose $i = a^\alpha . b^\beta . c^\gamma \ldots$, the order of this primitive part in the variables x, y, z, it is easy to see, will be $a^{2(\alpha-1)} . b^{2(\beta-1)} . c^{2(\gamma-1)} \ldots \{(a^2-1)(b^2-1)(c^2-1)\ldots\}$, which may be called the quadri-totient to i, and is the product of two factors, one the totient of i and the other what that totient becomes when $+1$ is substituted throughout for -1 in its expression, and which, if a name were needed for it, might be called the contra-totient.

The number of proper, or primitive, i^{th} sub-derivatives of any point of inflexion will thus be the quadri-totient of i (just as the number of primitive i^{th} roots of unity is the totient), and the total number of pertactile points of the i^{th} grade, 9 times the quadri-totient of i.

It is easy to see that the points corresponding to the non-primitive factors of X, Y, Z satisfy, but in an *improper* manner, the conditions of the question. For, if i' is any sub-multiple of i (say $i' = \frac{i}{\delta}$) and P' is an i'^{th} subderivative of a point of inflexion, through P' may be drawn δ curves each of the order i' (constituting an improper curve of the order i), each passing through $3i'$ consecutive points, and consequently their *ensemble* passes through $\delta . 3i'$ or $3i$ consecutive points. We have now obtained the generalization of the theorem of which the enumeration of the points of inflexion and Plücker's points constitute the two first steps, and it is very easy to calculate the number of pertactile points N of any given grade i. Thus for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 \ldots$

$$\frac{N}{9} = 1, 3, 8, 12, 24, 24, 48, 48, 72, 72, 120, 96 \ldots$$

The calculation is facilitated by the remark that if i, j are prime to each other, the number of $(ij)^{\text{th}}$ subderivatives to any one point of inflexion is the product of the number of i^{th} by the number of j^{th} subderivatives; the quadri-totient obeying the same law as the totient in this particular.

If i is the grade of the pertactile point P, so that $P_1 = P_{3i-1}$, then P_i is an inflexion, and P_{3i} is I, the original inflexion. Moreover

$$P_1 = P_1, P_2 = P_{3i-1}, P_2 = P_{3i+1}$$

$$P_2 = P_1, P_1' = P_1, P_{3i-1} = P_{3i-2} \text{ and also } = P_2, P_4 = P_{3i-2}, P_4 = P_{3i+2}$$

$$P_4 = P_2, P_2 = P_2, P_{3i-2} = P_{3i-4} \text{ and also } = P_2, P_{3i+2} = P_{3i+4}, \text{ and so on.}$$

And again, $P_3' = P_3 I = P_3, P_{3i} = P'_{3i+3}$, and therefore $P_3 = P_{3i+3}$;

and $\qquad P_{3i-3} = P'_{3i+3}, P_6 = P'_3, P_6 = P'_3$ whence $P_3 = P'_{3i-1}$;

$P_6 = P'_3, P'_3 = P'_{3i+3}, P'_{3i+3} = P_{6i+6}$ and also $= P_{3i-3}, P_{3i-3} = P'_{6i-6}$.

Thus in general, $P_{3r+1} = P_{3i\pm(3r+1)}$; $P_{3r-1} = P_{3r\pm(3r-1)}$

and $$P_{3r} = P_{3i+3r} = P'_{3i-3r}$$

Thus the natural scale $\qquad P_1\, P_2\, P_4\, P_5 \ldots .$

and the completed scale $\qquad \begin{cases} P_1\, P_2\, P_3\, P_4\, P_5\, P_6 \ldots . \\ P'_1\, P'_2\, P'_3\, P'_4\, P'_5\, P'_6 \ldots . \end{cases}$

are each of them periodic, the period of the indices being $3\,i$. We may, accordingly, describe pertactile by the simpler name of periodic points. Every complete set of periodic points forms a closed system. By a complete set is to be understood the $9\,i^2$ subderivatives of the 9 points of inflexion, and by a closed system is to be understood one such that every connective and tangential of the points which it contains is itself a point of the system. According to what law such closed system may be resolved into partial closed systems must form the subject of further inquiry. When $i = 2$, the complete closed system of 36 points we know is resoluble into nine closed systems, each containing one point of inflexion and its three collinear anti-tangentials, and also, in four different ways, into three closed systems, each containing a collinear set of inflexions and their three sets of anti-tangentials.

We are now in a position to solve the problem of in-and-exscribed k-laterals.

Suppose $k = 3$, then $2^3 + 1 = 3\,i$ where $i = 3$, and the point P_1 will coincide with the point P_8, provided P_3 is a point of inflexion. So that the apices of the in-and-exscribed triangles are the 81 points which satisfy the equation $P_8 = P'_6$, of which 9 will correspond to the points of inflexion and 72 remaining over will give 24 finite triangles. If we denote by p, p', p'' three consecutive points in a straight line at any point of inflexion, $pp', p'p'', p''p$ form an infinitesimal triangle degenerating into a straight line, and this furnishes an improper solution of the question.

Calling $M, N, xyz\Omega$ the co-ordinates of P_3 when $P_1 = x, y, z$, the 72 points are given by combining the equation $MN\Omega = 0$ with the equation to the curve.

If $k = 4$, we make $2^4 - 1 = 3\,i$ where $i = 5$, and if $P_1 = P_{3i-1}$, we have also $P_1 = P_{3i+1}$; and the apices of the quadrilateral are found by making P_i, i. e. P_5, a point of inflexion.

The general form of P_5 being xU, yV, zW, the proper subderivatives P_5 result from $UVW = 0$ combined with the equation to the cubic, and there result $\frac{9\,(25-1)}{4}$, i. e. 54 in-and-exscribed quadrilaterals.

Each point of inflexion may still be regarded as yielding an improper solution of the question, since pp', $p'p''$, $p''p'$, $p'p$ may be viewed as a degenerate infinitesimal quadrilateral.

So when $k = 5$, making $2^5 + 1 = 3i$, $i = 11$; and there will result $\frac{9(11^2-1)}{5} = 216$ in-and-exscribed pentagons.

Likewise, since $\frac{2^7+1}{3} = 43$, there result $9\,\frac{43^2-1}{7}$, i. e. 9.264 or 2376 in-and-exscribed heptagons.

Let us now consider a case of k a composite number, and to fix the ideas, suppose $k = 15$. Make $\frac{2^{15}+1}{3} = i$, then $i = 10923$. $\frac{2^{15}+1}{3}$, *by virtue of its form*, contains the factors $\frac{2^3+1}{3}$ and $\frac{2^5+1}{3}$, i. e. 3 and 11, and is in fact equal to 3 . 11 . 331. P_i will therefore be of the form $xU_3U_{11}\mathbf{U}$, $yV_3V_{11}\mathbf{V}$, $zW_3W_{11}\mathbf{W}$ (xU_3, yV_3, zW_3 corresponding to P_3, and xU_{11}, yV_{11}, zW_{11} to P_{11}).

Accordingly $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ will each be of the degree $(3\,.\,11\,.\,331)^2 - 3^2 - 11^2 + 1$, and the equation $\mathbf{UVW} = 0$, combined with the equation to the curve, will give the apices of the in-and-exscribed quindecagons, not including the improper solutions due to the points of inflexion, nor those due to the apices of the in-and-exscribed triangles or pentagons, which, in a certain but improper sense, each belong to the case of quindecagons. The number of apices of the proper quindecagons will therefore be $9\,[(3\,.\,11\,.\,331)^2 - 3^2 - 11^2 + 1]$, comprising sub-inflexions of several grades, as follows: $9\,(331^2 - 1)$ of the 331[th] grade, $9\,(3^2 - 1)\,(11^2 - 1)$ of the 33[d] grade, $9\,(3^2 - 1)\,(331^2 - 1)$ of the 993[d] grade, $9\,(11^2 - 1)\,(331^2 - 1)$ of the 3641[th] grade, and $9\,(3^2 - 1)\,(11^2 - 1)\,(331^2 - 1)$ of the 10923[d] grade.*. The above number of apices may be written $9\,[11^2 3^2 (331^2 - 1) + (3^2 - 1)\,(11^2 - 1)]$, so that the number of quindecagons is $9\,[11^2\,.\,3^2\,.\,22\,.\,33 \;+ 8^2]$.

It may be noticed that the primitive algebraical factor of $2^{15} + 1$, viz. 331, is a prime number. But the primitive part of $2^k - 1$ (k being even) or $2^k + 1$ (k being odd), i. e. $2^k - 1$ or $2^k + 1$ stripped of its obligatory factors dependent algebraically on the prime factors of k, may be a composite number.

Thus, let us suppose $k = 9$, the problem being that of finding the nature and number of the in-and-exscribed nonagons. Here $i = \frac{2^9+1}{3} = 171$, $2^9 + 1$ having, besides the obligatory factor $2^3 + 1$ due to its algebraical form, the *two* factors 3 and 19.

* It is obvious that any derivative of an inflexion is itself an inflexion. For instance, if J is an inflexion, J_3 is the same as J, and J_2 (viz. J', J_2) is either $J\,.\,J_2$ i. e. J, or $(I, J), J_2$ i. e. $(I, J), J$ i. e. I (I being some other point of inflexion). Hence if $P_{\frac{i}{\delta}}$ is an inflexion, P_i is also an inflexion.

Taking each divisor of 171, viz. $3, 9, 19, 57, 171$, we see that the 3^d, 9^{th}, 19^{th}, 57^{th}, and 171^{th} subderivatives of the nine points of inflexion will each of them be an apex of an in-and-exscribed nonagon. Of these, the 3^d subderivatives, and they only, give improper solutions of the problem, they being the apices of the in-and-exscribed triangles. Hence the aggregate of proper apices and the corresponding nonagons separate into four distinct groups, corresponding to the primitive subderivatives of the $9^{th}, 19^{th}, 57^{th}$, and 171^{th} grades respectively, of the inflexions. The number of the nonagons belonging to the several groups will be the quadritotients of $9, 19, 57, 171$, i.e. $9^2-9, 19^2-1$, $(19^2-1)(3^2-1), (9^2-9)(19^2-1)$ respectively, i.e. 171^2-9, exactly the same as if 57 had been a prime number N, in which case the $(3N)^2$ subderivatives of an inflexion of the grade $3N$ would be subject to the deduction of $9-1$ for in-and-exscribed triangles, and 1 for the point itself.

To make more clear the distinct solutions of which the problem of in-and-exscription of a k-lateral in general admits, consider the case of $k = 8$. Here

$$i = \frac{2^8-1}{3} = \frac{2^4-1}{3}(2^4+1) = 85.$$

The first factor (the one algebraically contained in i) is 5 and the primitive algebraical factor is 17. The total number of octagonal apices will be $9(85^2-5^2)$, the number 5^2 corresponding to the points of inflexion and the in-and-exscribed quadrilaterals. These 255^2-15^2 apices will consist of points of the form $I_{\frac{1}{17}}$ and $I_{\frac{1}{85}}$, the number of the former being $9(17^2-1)$ and of the latter $9(17^2-1)(15^2-1)$.

It is easily seen that, in general, the number of apices of in-and-exscribed k-laterals is nine times the *functional totient* of $\left(\frac{2^k-\bar{1}^k}{3}\right)^2$, or, what is the same thing the number of apices is the functional totient of $(2^k-\bar{1}^k)^2$, as previously stated in Note to Proem in the last number of the *Journal;* the number of k-laterals is, of course, the number of apices divided by k. For instance, we thus have for the number of apices of quindecagons, nonagons, and octagons, respectively,

$$(2^{15}+1)^2-(2^3+1)^2-(2^5+1)^2+(2^1+1)^2,$$

$$(2^9+1)^2-(2^3+1)^2, \quad (2^3-1)^2-(2^4-1)^2,$$

as found above.

Since i is odd, every divisor of i will necessarily be so too. Conversely, it is easy to prove that every odd subderivative of a point of inflexion is an apex of an in-and-exscribed polygon, and to determine the number of its sides. For let i, any odd number, be given, and let k be the least number which will satisfy

the condition that $2^k - \bar{1}^k$ shall be a multiple of $3i$, then the sub-inflexions of the i^{th} grade will be the apices of an in-and-exscribed k-lateral. I give, in the annexed table, the values of k corresponding to a given value of i, which, of course, are unique; whereas to a given value of k, in general, several values of i will correspond.

i	3	5	7	9	11	13	15	17	19	21	23	25	27
k	3	4	6	9	5	12	12	8	9	6	22	20	27

to which may be subjoined the reciprocal table

$k = 3$	$i = 3$
$k = 4$	$i = 5$
$k = 5$	$i = 11$
$k = 6$	$i = 7, 21$
$k = 7$	$i = 43$
$k = 8$	$i = 17, 85$
$k = 9$	$i = 9, 19, 57, 171$
$k = 10$	$i = 31, 341$
$k = 11$	$i = 683$
$k = 12$	$i = 13, 15, 35, 39, 65, 91, 195, 273, 455, 1365.$

To illustrate the way in which this table is formed, take the case of $k = 12$; then $\frac{2^{12}-1}{3} = 3 \cdot 5 \cdot 7 \times 13$ where 3 belongs to $k = 3$, 5 to $k = 4$, 7 to $k = 6$; the values of i are found by taking the divisors of 1365, except those which are found set against $k = 3$, $k = 4$, $k = 6$, i. e. 3, 5, 7, 21.

The successive tangentials of any even-graded inflexional subderivative as $2^q i$, where i is odd, will evidently consist of a chain of q points attached to the ring formed by the apices of an in-and-exscribed polygon of k sides, where k is the least number which makes $2^k \pm 1$ divisible by $3i$.

In all cases (since k is to have the minimum value which makes $\frac{2^k \pm 1}{3}$ contain i) $2k$ must be $\tau(3i)$ or a submultiple of it, so that, if $i = 3^q j$, k is either $3^q \tau j$ or a submultiple of it; when $i = 3^q$, since the cyclotomic functions of the first species $\chi_3 2, \chi_{3^2} 2, \ldots \chi_{3^q} 2$ can only contain the first power of the intrinsic divisor 3, it follows that $k = 3^q = i$, as is seen in the table to be the case for $i = 3, 9, 27$; or, in other words, a 3^qth subderivative of a point of inflexion is an apex of an in-and-exscribed polygon of 3^q sides.

It may be as well to mention again here, by way of a remind, that the number of in-and-exscribed k-laterals whose apices are i^{th} subderivatives of the

inflexions, is always the k^{th} part of nine times the quadritotient of i; when $i = 3^q$ this number will be $\frac{1}{3^{q-2}}\{3^{2q} - 3^{2q-2}\}$, i. e. $3^{q+2} - 3^q$, being thus $24, 72$, 216, etc., for triangles, nonagons, eikosiheptagons, etc.

TITLE 5. — *An Exact Proof of the Scalar Law of Squares.*

I will now give an exact proof of the law that the order in the variables of P_n is n^2 in regard to the co-ordinates of P, and furthermore that the co-ordinates when $i = 3m \pm 1$ are of the form xU, yV, zW, and when $i = 3m$ are of the form $M, N, xyz\Omega$; x, y, z being the co-ordinates of the primitive P_1 and U, V, W, M, N, Ω quantics in x^3, y^3, z^3. Of course the order of a point means the order of its system of co-ordinates *expressed in its lowest terms*, that is to say when the values of the three co-ordinates have no common measure, and consequently the co-ordinates of any *two* of them are relatively prime in an algebraical sense, as follows from the equation $X^3 + Y^3 + Z^3 + kXYZ = 0$.

The law to be established comprises, it will be seen, two elements,— one numerical, the *rule of squares*; the other formal, containing two rules, one regarding the *distribution* of x, y, z between the co-ordinates, the other the quanticity of the parts not multiplied by x, y, z or xyz in respect to x^3, y^3, z^3.

Let us suppose that the law is true up to n inclusive. I shall show that it is true up to $2n$ inclusive.

1° For the case of $2i$ where $i \leqq n$.

Let X, Y, Z be the system of co-ordinates to P_i in its lowest terms; then, by the law of compound derivation, P_{2i} is $X(Y^3 - Z^3), Y(Z^3 - X^3), Z(X^3 - Y^3)$.

If these regarded as functions of X, Y, Z had any common measure X, Y or $X, Z^3 - X^3$ would have a common measure. Hence X, Y, Z would all have a common measure. Nor can they have any common factor F, a function of x, y, z. For in that case, when $F = 0$, we should have

$$Y^3 - Z^3 = 0\,,\ Z^3 - X^3 = 0 \text{ or } X^3 = Y^3 = Z^3,$$

and the arbitrary parameter k would be $-3 \cdot 1^{\frac{1}{3}}$, so that the cubic would become a triplet of straight lines, a supposition which falls outside the pale of the question.

Hence P_{2i} will be of four times the order of P_i, and therefore, by hypothesis, of the order $4i^2$, i. e. $(2i)^2$. Also, obviously, the form xU, yV, zW or $M, N, xyz\Omega$ (as the case may be) which exists for i is maintained for $2i$, which is or is not divisible by 3 according as i is or is not so divisible.

2° Let the index be any odd number less than $2n$.

I shall first establish a Lemma concerning the co-ordinates given by my

formulæ for the connectives of P, Q and P', Q, where P' is the opposite to P in respect to a given point of inflexion (say $x=1$, $y=-1$), and $x^3+y^3+z^3+kxyz=0$ is the equation to the cubic.

The connectives of $\begin{matrix}(u, v, w) \\ (u', v', w')\end{matrix}$ and of $\begin{matrix}(v, u, w) \\ (u', v', w')\end{matrix}$

are represented respectively by

$$\left.\begin{matrix} vwu'^2 - v'w'u^2 \\ wuv'^2 - w'u'v^2 \\ uvw'^2 - u'v'w^2 \end{matrix}\right\} \text{ and } \left\{\begin{matrix} uwu'^2 - v'w'v^2 \\ wvv'^2 - w'u'u^2 \\ vuw'^2 - u'v'w^2, \end{matrix}\right.$$

the 3^d co-ordinate being the same in both systems, which, of course, remain to be reduced to their simplest terms, being at present each of the order $2i^2+2j^2$.

I say that the same quantity F cannot divide each of the two sets of quantities when u, v, w; u', v', w' are derivatives, one of an even, the other of an odd grade of the same point on the cubic.

For, if so, let $F=0$; then each quantity in the two systems becomes zero.

Call $\quad \dfrac{u}{w}, \dfrac{v}{w}; \dfrac{u'}{w'}, \dfrac{v'}{w'}, \quad r, s; r', s'$ respectively.

Then $\quad (1) \dots\dots sr'^2 - s'r^2 = 0 \qquad rr'^2 - s's^2 = 0 \dots\dots (3)$

$\qquad\quad (2) \dots\dots rs'^2 - r's^2 = 0 \qquad r'r^2 - ss'^2 = 0 \dots\dots (4)$

$\qquad\qquad\qquad (5) \dots\dots rs = r's'.$

Writing $r^3=R$, $s^3=S$, $r'^3=R'$, $s'^3=S'$; $5, (3, 4), (1, 2)$ respectively give $RS=R'S'$, $RR'=SS'$, $R'S=RS'$. The second and third of these combined give $R^2=S^2$, $R'^2=S'^2$ and the first and second combined give $R'^2=S^2$. Hence, $R^2=R'^2=S^2=S'^2$, and consequently the original equations $(1), (2), (3)$ give $S=S'$, $R=R'$, $R=S'$ or $r^3=s^3=r'^3=s'^3$.

Let $r=\alpha s$, $r'=\beta s'$, $s=\gamma s'$. Then $\alpha^3=\beta^3=\gamma^3=1$, and all the equations $(1), (2), (3), (4), (5)$ will easily be found to be satisfied when (and only when) $\alpha=\beta\gamma$.

The equations $r^3=s^3$, $r'^3=s'^3$, i. e. $u^3=v^3$, $u'^3=v'^3$, imply that the points P, Q are two either distinct or identical anti-tangentials to the same point of inflexion $x=1$, $y=-1$. I say that this is impossible when P, Q are derivatives of the degrees i, j of the same point U on the curve, if $i+j$ is an odd number. It must be noticed that P and Q (two Plückerian points belonging to the same point of inflexion I) are identical with P' and Q' respectively.

Any even-degreed derivative of P or Q is I, and any odd-degreed derivative is the same point P or Q over again.

Let now $i\mu - j\nu = 1$. Then $U = U_{i\mu - j\nu}$ will be (without regard to the modulus 3) the connective of $U_{i\mu}$ and $U_{j\nu}$, because we may substitute at will U_i' for U_i and U_j' for U_j. But $U_{i\mu}$ and $U_{j\nu}$, if μ, ν be both odd, will be U_i and U_j over again, or if μ, ν be one odd and the other even, will be I and one of the two Plückerian points.

Hence U is the connective of I and a Plückerian point, or else of two Plückerians which are identical, or of two Plückerians (both appurtenant to I) which are distinct.

In the 1st and 3d cases, then, U is a Plückerian, in the 2d case a point of inflexion. But every derivative of a point of inflexion is a point of inflexion, and every even-degreed derivative of a Pluckerian is also a point of inflexion; but by hypothesis (since one of the two numbers i, j is even) an even-degreed derivative of U is a Plückerian, which is self-contradictory. Hence, it follows that the expressions given by *my* formulæ for the connectives of P_i, P_j and P_i', P_j when $i+j$ is odd, say P, Q, R; P', Q', R, cannot have a common factor; so that if M is a common measure of P, Q, R and M' of P', Q', R', M is relatively prime to M'.

Let ϕ, ψ, ω be always understood to mean $\phi(x^3, y^3, z^3)$, $\psi(x^3, y^3, z^3)$, $\omega(x^3, y^3, z^3)$; let $(\mu), (\nu)$ be understood to mean the prime systems of co-ordinates u, v, w; u', v', w' which represent μ, ν (μ and ν being numbers, accented or unaccented, representing derivatives to the index μ and ν) let $[\mu, \nu]$ represent the unreduced system of the co-ordinates of the connective of μ, ν, viz. $v'w'u^2 - vwu'^2$, $w'u'v^2 - wuv'^2$, $u'v'w^2 - uvw'^2$; (μ, ν) the above system *reduced* by elimination of the greatest common measure of its terms.

If $(\mu), (\nu)$ are each of the form $x\phi, y\psi, z\omega$, $[\mu, \nu]$ is of the form $x^2yz\phi_1$, $xy^2z\psi_1$, $xyz^2\omega_1$, but $[\mu', \nu]$, i. e. the unreduced connective of $y\psi, x\phi, z\omega$; $x\phi', y\psi', z\omega'$, is of the form $z\phi_1, z\psi_1, xyz^2\omega_1$.

Again, if (μ) is of the form $x\phi, y\psi, z\omega$ and (ν) of the form $\phi_1, \psi_1, xyz\omega_1$, $[\mu', \nu]$, the unreduced connective of the systems $y\psi, x\phi, z\omega$ and $\phi_1, \psi_1, xyz\omega_1$, is easily seen to be of the form $zx\Phi, zy\Psi, z^2\Omega$.

Furthermore, the order in the variables of (p') is obviously the same as that of (p).

Now it has been shown under Title 2 that

$$6i - 1 = \overline{3i-1}', 3i \qquad 6i - 5 = \overline{3i-3}', \overline{3i-2}' \qquad 6i - 3 = \overline{3i-2}', 3i - 1.$$

If, then, $(3i)$ and $(3i-3)$* are of the form $\phi, \psi, xyz\omega$, and $(3i-2), (3i-1)$

* $\overline{3i-3}'$ will obviously be of the same form as $3i - 3$.

each of the form $x\phi, y\psi, z\omega$, it follows that $[6i-1]$ and $[6i-5]$ will be of the form $zx\phi, zy\psi, z^2\omega$ and $[6i-3]$ of the form $z\phi, z\psi, xyz^2\omega$.

The above inference suffices to show that, if, for all values of $3\mu \pm 1$ and 3μ up to n inclusive, it be true that $(3\mu \pm 1)$ is of the form $x\phi, y\psi, z\omega$ and of the order $(3\mu \pm 1)^2$, and (3μ) is of the form $\phi, \psi, xyz\omega$ and of the order $(3\mu)^2$; then the same will be true up to $2n$ inclusive.

That this is true for even values not exceeding $2n$ appears from what has been already shown. Confining, then, our attention to odd numbers less than $2n$; these must be representable by $6i-5, 6i-3$ or $6i-1$, and by hypothesis the form of each of the systems $(3i), (3i-1), (3i-2), (3i-3)$ fulfils the conditions of the last paragraph but one; consequently the form of $[6i-5]$, $[6i-3]$, $[6i-1]$ will be $zx\phi, zy\psi, z^2\omega$; $z\phi, z\psi, xyz^2\omega$; $zx\phi, zy\psi, z^2\omega$, i. e. in every case the factor z will be contained in each term of the system $[\overline{\iota-1}', \iota^?]$, which represents an unreduced system of co-ordinates of the point $2\iota-1$, the mark of interrogation signifying a blank or an accent as the case may be.

But either the point 1 or the point 1′ will, in every case, correspond to the connective obtained by changing $\overline{\iota-1}'$ into $\iota-1$;* moreover, the unreduced system of co-ordinates to that connective will have the third term, say π, in common with the unreduced system to $2\iota-1$ above mentioned.

This contrary system we know must have the common factor $\frac{\pi}{z}$ because 1 and 1′ are denoted by x, y, z; y, x, z respectively. Hence the unreduced system for $2\iota-1$ can have no other common factor except z, which they have been shown to have; since, were it otherwise, the *two* contrary systems would have some quantity contained in $\frac{\pi}{z}$ for a joint common measure, which has been proved to be impossible.

Hence, the form of $(2\iota-1)$ is $x\phi, y\psi, z\omega$ or $\phi, \psi, xyz\omega$ according as $2\iota-1$ is not or is divisible by 3, and its order is in all cases $2\overline{\iota-1}^2 + 2\iota^2 - 1$, i. e. $(2\iota-1)^2$.

Hence the form-law of distribution of the simple powers of the variables x, y, z and of the quanticity in x^3, y^3, z^3 of the multipliers of x, y, z or of $1, 1, xyz$, as well as the numerical law that the order of any derivative is the square of its index, will be true up to $2n$ inclusive if true up to n inclusive; and being true for $n=1$, is true universally.

As a corollary we may now do away with the restriction of $i+j$ being odd, and affirm that in all cases (the futile one of $i=j$ alone excepted), if the reduced

* For, on consulting Title 2, it will be found that *in every case*, if the arithmetical value of the index of P_i, P_j is $i \pm j$, that of P_i', P_j is $(\iota \mp j)^2$.

system of co-ordinates to the connective of P_i, P_j be F, G, H and to that of P_i', P_j be F', G', H', then the unreduced system expressing those connectives given by my formulæ of connection will be $H'F$, $H'G$, $H'H$; HF', HG', HH', respectively; for the two systems of unreduced co-ordinates (each of the order $2i^2+2j^2$) contain, one of them a common factor of the order $(2i^2+2j^2)-(i-j)^2$ i. e. $(i+j)^2$, the other a common factor of the order $(2i^2+2j^2)-(i+j)^2$ i. e. $(i-j)^2$, and these two factors being prime to each other, their product must be contained in the term common to the two systems, and being of the same order $(i+j)^2+(i-j)^2$ as that common term, must be equal to it.

Hence, if π be the common unreduced term, and H, H' the two reduced terms, we must have $\pi = \frac{\pi}{H} \cdot \frac{\pi}{H'}$ or $\pi = HH'$, as was to be shown.

As a matter rather of curiosity than of real importance I will state the analogous law when the connective and cross-connective between two derivatives is expressed by Cauchy's formulæ instead of my own. These formulæ, it will be remembered, give for the co-ordinates of the connective of u, v, w; u_1, v_1, w_1 the minor determinants of the matrix

$$\begin{vmatrix} vw_1 - v_1w \;;\; w_1u - wu_1 \;;\; uv_1 - u_1v \\ uu_1 \quad ; \quad vv_1 \quad ; \quad ww_1 \end{vmatrix}$$

If, now, the prime system of co-ordinates to the connectives of P_i, P_j; P_i', P_j be denoted as before by F, G, H; F', G', H', I find by calculation that the Cauchian formulæ will present these two systems under the unreduced forms

$$(F'+G')\,F\,,\;(F'+G')\,G\,,\;(F'+G')\,H$$
$$(F+G)\,F'\,,\;(F+G)\,G'\,,\;(F+G)\,H',$$

between which there is no common term; and consequently, had I not discovered my own simpler formulæ, the method of proof of the Law of Squares which I have employed would have been inapplicable, and it is not easy to see what other strict method of proof could have taken its place.

I have thus accomplished the very difficult task of proving a negative, in this instance the non-existence of *latent* common factors to the co-ordinates of the connective of any two given derivatives. I might have founded a much easier proof of the Law of Squares upon Mr. Franklin's geometrical solution of the problem of finding the number of in-and-exscribed k-laterals to a cubic (if one could feel quite assured *à priori* of the strict logic of the process*) as follows: He

* In that solution the apices are found as the intersections of the cubic with another curve. Certain of these intersections are seen from geometrical considerations to count twice, and others three times; but while we have no reason to suppose any further cause of reduction, the non-existence of such cause is not proved.— F. F.

has virtually found (*vide* last number of the Journal) that the number of apices of the in-and-exscribed k-laterals of *every kind* [and not excluding the points of inflexion] is $(2^k - \overline{1}^k)^2$. If, then, $2^k - \overline{1}^k = 3i$, it follows from what has been shown in the preceding pages, that the order of P_i in the co-ordinates of P is $\frac{1}{9}(3i)^2$, i. e. i^2.

Let now i' be any number whatever, and τ the totient of $3i'$; then τ is even, and, by Fermat's Theorem, $2^\tau - \overline{1}^\tau = 3i'i''$.

Hence, if μ', μ'' are the orders of $P_{i'}$, $P_{i''}$ respectively, the law of compound derivation will suffice to lead to the conclusion that $\mu'\mu''$ will be the order of $P_{i'i''}$, and accordingly $\mu'\mu'' = i'^2 i''^2$; but $\frac{\mu'}{i'^2}$, $\frac{\mu''}{i''^2}$, it has been proved under a preceding Title, are neither of them greater than unity: hence each of them is equal to unity, and i'^2 is the order of $P_{i'}$, as was to be shown.

ADDENDUM ON THE DEGORDER OF THE DERIVATIVES TO A POINT ON A CUBIC IN THE NATURAL SCALE.

Let n be any number not divisible by 3. The n^{th} derivative, it has been proved, is of the order n^2 in the variables. It remains to determine its *degree in the coefficients.*

When $n = 2$ we know that the degorder is $[4 ; 4]$, each new co-ordinate being one of the minors of the rectangular matrix

$$\begin{vmatrix} \frac{dU}{dx} & \frac{dU}{dy} & \frac{dU}{dz} \\ \frac{dH}{dx} & \frac{dH}{dy} & \frac{dH}{dz} \end{vmatrix},$$

where U is the cubic and H its Hessian.

Suppose ν to be the degree in the coefficients of the n^{th} derivative. Then the degree of the $(2n)^{\text{th}}$ derivative regarded as the second of the n^{th} will be $4\nu + 4$, and regarded as the n^{th} of the second will be $n^2 . 4 + \nu$, and these two must be equal. Hence $3\nu = (n^2 - 1) 4$ or $\nu = \frac{4}{3}(n^2 - 1)$.

Hence the degorder of any n^{th} derivative in the natural scale is $\left[\frac{4n^2 - 4}{3} ; n^2\right]$. If we substitute the co-ordinates of this derivative in the given cubic U, the result must be of the form $U . R$ and will be of the degorder $[1 + 4n^2 - 4 ; 3n^2]$. Hence R is of the degorder $[4n^2 - 4 ; 3n^2 - 3]$. If the well-known covariant of the degorder $[12 ; 9]$ be called J, R is of the same degorder as $J^{\frac{n^2-1}{3}}$, and possibly

may be found to be identical with it. To corroborate the validity of the determination of the degorder of the n^{th} derivative, we may proceed as follows:

Imagine, at first, the cubic to be reduced to the canonical form $x^3 + y^3 + z^3 - 3\,kxyz$. The connective of P_1, P_2 in its reduced form is x, y, z; but in its unreduced form and prior to all simplification, will, by virtue of the theory (Titles 1 and 5), be of the form Mx, My, Mz where

$$M = x^3y^6 + y^3z^6 + z^3x^6 + x^6y^3 + y^6z^3 + z^6x^3 - 6\,x^3y^3z^3$$
$$+ kxyz\,(x^6 + y^6 + z^6 - y^3z^3 - z^3x^3 - x^3y^3)\,;^*$$

consequently M expressed (as I shall hereafter suppose) in terms of the original coefficients and variables, will be of the degorder $[9\,;\,9]$: for Mx, My, Mz are of the degorder $[1 + 2\,.\,4\,;\,2\,(1 + 4)]$, i. e. $[9\,;\,10]$.† Also the degorder of P_4 will be $[4 + 4\,.\,4\,;\,16]$, i. e. $[20\,;\,16]$.

Suppose now we wish to find the degorder of P_5.

The unreduced connective of P_1, P_4 will be of the form MX, MY, MZ, where X, Y, Z are the reduced co-ordinates and M is exactly the same thing as before. The degorder of the unreduced co-ordinates will be $[1 + 2\,.\,20\,;\,2\,(1 + 16)]$, i. e. $[41\,;\,34]$; and consequently, subtracting $[9\,;\,9]$, the degorder of X, Y, Z will be $[32\,;\,25]$, i. e. $\left[4\,\frac{5^2-1}{3}\,;\,5^2\right]$.

So, again, to find P_7 we may regard it as the connective of P_2, P_5. The unreduced degorder of P_7 will thus be seen to be $[1 + 2\,(4 + 32)\,;\,2\,(4 + 25)]$, i. e. $[73\,;\,58]$, and subtracting, as before, $[9\,;\,9]$, the degorder of the reduced co-ordinates of P_7 becomes $[64\,;\,49]$ i. e. $\left[4\,\frac{7^2-1}{3}\,;\,7^2\right]$, agreeable to what has been previously found; and so, in general, supposing the degrees of P_μ and $P_{\mu+3}$ in the coefficients to be $4\,\frac{\mu^2-1}{3}$ and $4\,\frac{(\mu+3)^2-1}{3}$, the unreduced degree of $P_{2\mu+3}$ will be $1 + 8\left\{\frac{\mu^2-1}{3} + \frac{(\mu+3)^2-1}{3}\right\}$, from which subtracting 9, the reduced degree becomes $8\left\{\frac{2\,\mu^2+6\,\mu+4}{3}\right\}$, which is the same thing as $4\left\{\frac{(2\,\mu+3)^2-1}{3}\right\}$, as ought to be the case. There is, therefore, no loophole for doubt left open as regards the degorder of any *natural* derivative to the index k (a number necessarily of the form $3\,i \pm 1$) being $[\frac{4}{3}\,(k^2 - 1)\,;\,k^2]$, a notable result!

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* It is worthy of remark that, if we make $U = 0$, so that $3\,kxyz$ becomes equal to $x^3 + y^3 + z^3$, the expression in the text for M gives $3\,M$ equal to the norm of $x + 1^{\frac{1}{3}}y + 1^{\frac{1}{3}}z$, namely, $(x^3 + y^3 + z^3)^3 - 27\,x^3y^3z^3$.

† In fact, M, as may easily be shown, is the covariant $\left[\Sigma\left(\frac{dU}{dy}\cdot\frac{dH}{dz} - \frac{dU}{dz}\cdot\frac{dH}{dy}\right)\frac{d}{dx}\right]^2 U$, in other words the symmetrical determinant of the 5th order formed by double-bordering the Hessian matrix with the differential derivatives of the Hessian and of the original cubic.

We are now in possession of a method for finding any natural derivative to the index n. If n is even, it may be derived immediately from the derivative to the index $\frac{n}{2}$. If n is odd, it must be of the form $2\mu + 3$ where μ is not divisible by 3.

Taking P as the initial point, P_μ and $P_{\mu+3}$ may be considered as known. Calling their co-ordinates X, Y, Z; X_1, Y_1, Z_1 respectively, and substituting $\lambda X + \mu X_1, \lambda Y + \mu Y_1, \lambda Z + \mu Z_1$ in the equation to the cubic, we shall obtain an equation of the form $\lambda^2\mu B + \lambda\mu^2 C = 0$. The unreduced co-ordinates of $P_{2\mu+3}$ will then be $CX - BX_1$, $CY - BY_1$, $CZ - BZ_1$, which will contain a common measure M of the degorder [9; 9], and $\frac{CX - BX_1}{M}$, $\frac{CY - BY_1}{M}$, $\frac{CZ - BZ_1}{M}$ will be the expression for the point $P_{2\mu+3}$ in its simplest terms.

More generally, if $n = 2\mu + 3i$, we may obtain, in like manner as above, the unreduced co-ordinates of the connective to P_μ, $P_{\mu+3i}$, and, by an easy calculation, it will be found that the new common measure will be of the degorder [$12i^2 - 3$; $9i^2$], and will be constant, i. e. independent of μ for any given value of i, and identical with the common measure to the unreduced co-ordinates of P_{3i+2} regarded as the connective to P and P_{3i+1}.

It is well worthy of remark that if X, Y, Z be the co-ordinates of any derivative, and ξ, η, ζ contragredient to x, y, z, $X\xi + Y\eta + Z\zeta$ will be an invariantive concomitant to the given cubic. This gives rise to a new series of reflexions, the development of which must be deferred to a more convenient occasion.*

* It is obviously a step towards the attainment of the desideratum of finding the general expression for any derivative in an explicit form, or, at all events, by explicit processes and without the necessity for division of the unreduced co-ordinates by a common measure. This latter, it should be observed however, by virtue of what is stated above, is always known *à priori*.

On the General Equations of Electro-magnetic Action, with Application to a New Theory of Magnetic Attractions, and to the Theory of the Magnetic Rotation of the Plane of Polarization of Light.

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CONTENTS.

In the last number of this Journal I gave what may be considered as the introduction to this paper, and the two should be read continuously.

II. *General Equations in an Extensive Conducting Medium.*

In thinking over the subject there discussed, and seeing how intimately rotation is connected with magnetic phenomena, and how, according to Maxwell's theory of electric displacement, there can be no such thing as an electric current which is not closed, I have been led to a number of curious theorems. And I have been further guided by the idea of Faraday on the conduction of magnetic lines of force, and by the well-known equations of vortex motion which apply to electric and magnetic phenomena. To free my mind from all complications of conductors and non-conductors, and to place magnetic action and electric conduction on the same footing, I conceive of an infinite conducting medium filling all space in the same manner as all space is filled with a conductor of magnetic lines of force, and consider the electro-magnetic action. Such a medium is often used in the theory of electric conduction, but I am not

aware that others have occupied themselves much with the magnetic action of currents in such a case. At least I believe the following theory to be new.

It is usual in the ordinary theory of magnetic action to conceive of the poles of the magnet as separated from each other, and to define the magnetic field at any point as the force acting on a unit pole at that point. As analogous to that, let us conceive of points in the medium at which electricity is either generated or destroyed, so that electric currents shall radiate either from or to the points. Let us call these points electric points, and let the same quantity of electricity stream from them whatever their position, and let the strength of the point be denoted by e, where $4\pi e$ is the quantity of electricity streaming *from* it. Let the strength of the magnetic poles be denoted by m, which will be $+$ for North polarity and $-$ for South. This system is perfectly symmetrical with the other, as $4\pi m$ lines of force stream from magnetic poles of strength m.

AXIOM. Two magnetic poles, two electric points, or a magnetic pole and an electric point, cannot exert force on each other except in the direction of the line joining them, for they are symmetrical around that line.

Thus to find the action between an electric point and a magnetic pole, consider any two current rays situated symmetrically with respect to the pole: the true magnetic action of these two will be zero, and hence the force to and from the point will be zero. But the ends of these rays may still have an unknown action. One end of these rays is at the electrical point, and the other on the surface of the sphere at an infinite distance: the action of the sphere is zero and hence there may be a force between an electrical point and a magnetic pole with which we are yet unacquainted, and whose existence we are almost unable to prove experimentally, seeing that we can hardly experiment on unclosed circuits. But there is no true magnetic action such as we are acquainted with. This conducts us to the following remarkable proposition, which is extremely useful.

PROP. The *true** magnetic action of any system of currents which can be generated in an unlimited medium by electric points is zero. Or, in other words, the magnetic action of any system of electric currents which are acyclic is zero.

In an unlimited medium, then, unclosed electric currents have no magnetic action. According to Maxwell's theory, unclosed circuits cannot exist; but on the ordinary theory, the discharge of conductors produces such currents.

* By *true* magnetic action I mean such magnetic action as we detect in *closed* circuits, and do not include the direct attraction or repulsion between an electric point and a magnetic pole, which we have seen above *may* exist in the case of unclosed circuits. When speaking of magnetic action, I shall generally mean *true* magnetic action.

Let us now conceive of a plus and a minus electric point placed very near each other. Any series of such points can have no magnetic action. But now let us join the points by a line, and suppose such an electro-motive force to exist along the line as to produce a current equal to the strength of one of the points. The current in this line can then have magnetic action; and as the whole circuit is closed, we can calculate it by the well-known formula for the element of a closed circuit. For the unknown direct action between the pole and the electric points will then vanish, because the currents are closed, and there are no ends to the lines of flow. So that the whole magnetic action in the medium will be that due to the current along this line alone. A current can only be produced along such a line by the action of an electro-motive force along it.

Hence we have the remarkable proposition, that in such a medium the action reduces itself to an action between magnets and *electro-motive forces*, instead of between magnets and ordinary *electric currents*. And all the equations for the magnetic action of such currents are much simplified.

To represent the conduction of currents along wires of any shape in such a medium, we have only to suppose the electro-motive force to be in the direction of the wires, and by a proper distribution of electro-motive force, all cases of conduction in limited media can be represented.

Let us again conceive of a plus and a minus magnetic pole near each other in a stream of electricity coming from electric points. They will evidently not have any tendency to a motion of translation in any direction. But now let the poles be joined so that the lines of force flow from one to the other, and the magnet would probably tend to move across the currents. Hence, reasoning as before, this kind of magnetic action cannot be explained, unless the lines of magnetic induction form closed circuits. We cannot speak definitely about this case, as we cannot experiment on anything but closed circuits. But the action seems probable.

From the analogy with the case of electric conduction, let us suppose such a force to exist between the two magnetic points as to cause the same number of lines of induction to pass between the points as flow out of either of them, and call this force a *magneto-motive force*. Such a force is proportional to what has hitherto been called the magnetization, and replaces with a definite mathematical idea the old indefinite idea of *coercive force*.

As thus all lines of magnetic induction and all electric currents are closed, we have to deal in either case with cycles and cyclic regions. Hence the equations of vortex motion should apply to these cases.

Again, when a potential is cyclic, we know very well that we can divide the

region into acyclic regions by diaphragms, and that the potential in each of these regions can be represented by integrals taken over the diaphragms enclosing it.

With the ideas thus introduced, all equations giving the relations of *electric currents* to *electro-motive force* are exactly similar to the equations giving the relations of *magnetic induction* to *magneto-motive force*. All Thomson's ideas with respect to lamellar and solenoidal distribution of magnetism thus apply to electro-motive force. If we conceived electro-motive force distributed within a certain region similar to a magnet and proportional to the magnetization, the electric currents at every point of space would be proportional to the magnetic induction in the case of the magnet, without any limitation as to the point being within or without the magnet.

Should the components of the electro-motive force satisfy the lamellar or solenoidal condition within the region, which condition is the same as the equation of continuity, then a single diaphragm enclosing the region will divide space into two acyclic regions, and the electric currents within and without the region can be represented by integrals over this surface. The distribution of electric points for the outside integration must be similar to that of the so-called surface distribution of magnetism on the magnet.

The magnetic action of an electric current is well known to be equivalent to that of a magnetic shell with its edge in the current. Such a magnetic shell merely consists of a sheet of some substance magnetized in a direction normal to the surface. In terms of the words used in this paper, it is a sheet of matter possessing a magneto-motive force in a direction normal to the surface.

As there must be a similar and analogous theorem in the other direction, let us introduce the idea of an electro-motive shell, that is, a shell of matter which possesses an electro-motive force in a direction normal to its surface. The potential of the electric currents from such a shell is evidently equal to the solid angle subtended by the shell, as in the analogous case in magnetism.

Let us now develop the reciprocal relations of electro-motive force and magneto-motive force in a conducting medium. Let us call certain small regions possessing electro-motive or magneto-motive forces, electro-motive or magneto-motive points: these are similar to elements of currents or magnets, ordinarily so called. We then have the following reciprocal theorems.

An electro-motive point tends to pass across the lines of magnetic force in a direction at right angles to both, and therefore a magnetic pole tends to revolve around the electro-motive point.

Reciprocally, a magneto-motive point tends to pass across the electric current in a direction at right angles to both, and therefore an electric point tends to revolve around the magneto-motive point.

Could we prove that an electric point tends to move with the electric current in the same manner as a magnetic pole tends to move in the dirêction of the lines of magnetic force, the reciprocity would be more exact.

The forces which act or may act between two electric points may be divided into two classes, the electrostatic and electromagnetic forces. The electrostatic forces are evidently the same in the given system as if we should gradually reduce the conductivity to zero, the potential remaining the same. Hence the force between two electric points of strengths m and m' will be

$$\frac{Kmm'}{\mu'^2 r^2},$$

where K is the specific inductive capacity and μ' is the conductivity.

The calculation of the electromagnetic forces depends upon the existence of a force of tension along the current which will be produced when the elements have some action on each other in the direction of their length. As we are unable to prove the existence of such a force, we are unable to say whether two electric points attract each other electromagnetically or not.

That the reciprocity is not perfect, we also prove as follows. Each magnetomotive element is equivalent to an element having an electric current around it, or rather an electro-motive force around it; in other words, the magnetomotive force is proportional to what Maxwell calls the *curl* of the electro-motive force. But the lines of magnetic force around an element of electro-motive force fill all space according to a well-known law, and are not merely packed closely around the element.

The electric potential due to an electro-motive force, A', in an element, $dxdydz$, is, for all points outside the element, as we shall see further on,

$$V = A' \frac{\cos\theta}{r^2} \, dxdydz,$$

and the magnetic force

$$\mathfrak{H} = - A' \frac{\sin\theta}{r^2} \, dxdydz;$$

where θ is the angle between the radius vector and the axis of the electromotive force. It is evident that these can be both calculated from the same function, for let us define a quantity, U, by the equation,

$$U = A' \frac{1}{r} \, dxdydz;$$

then

$$V = -\frac{dU}{dx} \quad \text{and} \quad \mathfrak{H} = \frac{dU}{dg},$$

where

$$g = \sqrt{y^2 + z^2}.$$

Proceeding in this way we can deduce the whole theory of vector potentials as applied to this case, but with this difference from the ordinary theory that the vector potential is given with reference to the *electro-motive* force rather than to *electric currents*.

But I prefer to treat the subject in a manner similar to vortex motion, and symmetrically. I shall first give the theory for electric conduction and for magnetism separately, and then introduce the electro-magnetic relations of the two. Having thus developed the old theory, I shall then consider the modification required by the newly discovered action.

Let a, b, c be the components of the magnetic induction.

Select three quantities, F, G, and H, to satisfy the equations

$$a = \frac{dH}{dy} - \frac{dG}{dz}$$

$$b = \frac{dF}{dz} - \frac{dH}{dx}$$

$$c = \frac{dG}{dz} - \frac{dF}{dy};$$

whence

$$\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} = 0.$$

These equations are satisfied by taking any three quantities L, M, and N, such that

$$F = \frac{dN}{dy} - \frac{dM}{dz}$$

$$G = \frac{dL}{dz} - \frac{dN}{dx}$$

$$H = \frac{dM}{dx} - \frac{dL}{dy};$$

whence

$$\frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} = 0.$$

Hence, putting

$$J = \frac{dL}{dx} + \frac{dM}{dy} + \frac{dN}{dz},$$

Let a', b', c' be the components of the electric current.

Select three quantities, F', G', and H', to satisfy the equations

$$a' = \frac{dH'}{dy} - \frac{dG'}{dz}$$

$$b' = \frac{dF'}{dz} - \frac{dH'}{dx}$$

$$c' = \frac{dG'}{dz} - \frac{dF'}{dy};$$

whence

$$\frac{da'}{dx} + \frac{db'}{dy} + \frac{dc'}{dz} = 0.$$

These equations are satisfied by taking any three quantities, L', M', and N', such that

$$F' = \frac{dN'}{dy} - \frac{dM'}{dz}$$

$$G' = \frac{dL'}{dz} - \frac{dN'}{dx}$$

$$H' = \frac{dM'}{dx} - \frac{dL'}{dy};$$

whence

$$\frac{dF'}{dx} + \frac{dG'}{dy} + \frac{dH'}{dz} = 0.$$

Hence, putting

$$J' = \frac{dL'}{dx} + \frac{dM'}{dy} + \frac{dN'}{dz},$$

we have

$$a = -\Delta^2 L + \frac{dJ}{dx}$$

$$b = -\Delta^2 M + \frac{dJ}{dy}$$

$$c = -\Delta^2 N + \frac{dJ}{dz}\cdot$$

Whence, if we choose some other quantity, χ, so that

$$\chi = \frac{1}{\mu}\iiint \frac{J}{r}\,dxdydz,$$

we may write

$$L = \frac{1}{4\pi}\iiint \frac{a}{r}\,dxdydz - \mu\frac{d\chi}{dx}$$

$$M = \frac{1}{4\pi}\iiint \frac{b}{r}\,dxdydz - \mu\frac{d\chi}{dy}$$

$$N = \frac{1}{4\pi}\iiint \frac{c}{r}\,dxdydz - \mu\frac{d\chi}{dz}\cdot$$

$$\text{Let } p = \frac{1}{r}\cdot$$

Substituting these quantities in the previous equations, we have, as χ disappears,

$$F = \frac{1}{4\pi}\iiint\left\{\frac{d\,(pc)}{dy} - \frac{d\,(pb)}{dz}\right\}dxdydz$$

$$G = \frac{1}{4\pi}\iiint\left\{\frac{d\,(pa)}{dz} - \frac{d\,(pc)}{dx}\right\}dxdydz$$

$$H = \frac{1}{4\pi}\iiint\left\{\frac{d\,(pb)}{dx} - \frac{d\,(pa)}{dy}\right\}dxdydz$$

$$F = \frac{1}{4\pi}\iiint\left\{p\left(\frac{dc}{dy} - \frac{db}{dz}\right) + c\frac{dp}{dy} - b\frac{dp}{dz}\right\}dxdydz$$

$$G = \frac{1}{4\pi}\iiint\left\{p\left(\frac{da}{dz} - \frac{dc}{dx}\right) + a\frac{dp}{dz} - c\frac{dp}{dx}\right\}dxdydz$$

$$H = \frac{1}{4\pi}\iiint\left\{p\left(\frac{db}{dx} - \frac{da}{dy}\right) + b\frac{dp}{dx} - a\frac{dp}{dy}\right\}dxdydz.$$

we have

$$a' = -\Delta^2 L' + \frac{dJ'}{dx}$$

$$b' = -\Delta^2 M' + \frac{dJ'}{dy}$$

$$c' = -\Delta^2 N + \frac{dJ'}{dz}\cdot$$

Whence, if we choose some other quantity, χ', so that

$$\chi' = \frac{1}{\mu'}\iiint \frac{J'}{r}\,dxdydz,$$

we may write

$$L' = \frac{1}{4\pi}\iiint \frac{a'}{r}\,dxdydz - \mu'\frac{d\chi'}{dx}$$

$$M' = \frac{1}{4\pi}\iiint \frac{b'}{r}\,dxdydz - \mu'\frac{d\chi'}{dy}$$

$$N' = \frac{1}{4\pi}\iiint \frac{c'}{r}\,dxdydz - \mu'\frac{d\chi'}{dz}\cdot$$

$$\text{Let } p = \frac{1}{r}\cdot$$

Substituting these quantities in the previous equations, we have, as χ disappears,

$$F' = \frac{1}{4\pi}\iiint\left\{\frac{d\,(pc')}{dy} - \frac{d\,(pb')}{dz}\right\}dxdydz$$

$$G' = \frac{1}{4\pi}\iiint\left\{\frac{d\,(pa')}{dz} - \frac{d\,(pc')}{dx}\right\}dxdydz$$

$$H' = \frac{1}{4\pi}\iiint\left\{\frac{d\,(pb')}{dx} - \frac{d\,(pa')}{dy}\right\}dxdydz$$

$$F' = \frac{1}{4\pi}\iiint\left\{p\left(\frac{dc'}{dy} - \frac{db'}{dz}\right) + c'\frac{dp}{dy} - b'\frac{dp}{dz}\right\}dxdydz$$

$$G' = \frac{1}{4\pi}\iiint\left\{p\left(\frac{da'}{dz} - \frac{dc'}{dx}\right) + a'\frac{dp}{dz} - c'\frac{dp}{dx}\right\}dxdydz$$

$$H' = \frac{1}{4}\iiint\left\{p\left(\frac{db'}{dx} - \frac{da'}{dy}\right) + b'\frac{dp}{dx} - a'\frac{dp}{dy}\right\}dxdydz.$$

These equations, as we have thus obtained them, do not contain any of the phenomena of electro-magnetism, but each series of equations is entirely separate from the other series. The only assumption we have made is that a, b, and c, as well as a', b', and c', are vector quantities which satisfy the equation of continuity at *every* point of space. And this we know to be true in magnetism, where we cannot separate the North from the South pole of a magnet, and in the case of electric currents, provided we also include the displacement currents. If we could separate the poles of a magnet, or if we consider the electric currents from the discharge of electrified bodies without the currents of displacement, there will always be certain points where the lines of force or the currents end, and where, therefore, the equation of continuity does not apply. Both the lines of force, then, and the electric currents, must, as we have shown, be cyclic; and they can have no real potential in general, though by the proper placing of diaphragms we may cause the space to be acyclic, and so may express the potential by a proper integration taken over the diaphragms. The value of the potential so obtained will then apply to all space which can be reached without passing through a diaphragm.

Hence the integrals which we have given can be expressed either as surface integrals over some surface or surfaces, or as volume integrals throughout some volume occupied by the sources of the lines of induction or the electric currents. And this last proposition is evidently true from a physical point of view.

Commencing with the electric currents, let us suppose that an electro-motive force, A', acts throughout the element $dxdydz$, in the direction of x. The electric currents will then stream through the cube in the direction of x, and pass out through one end to return through space to the other end, thus completing a cycle.

By enclosing the element within a surface, we divide cyclic space into two acyclic regions. Within the cube the current will evidently be, provided we take a proper unit for measuring the electro-motive force,

$$4\pi\mu' A'.$$

The amount passing out each end of the cube is therefore

$$4\pi\mu' A' dydz\,;$$

and the current potential at every point situated outside the cube and at a very great distance, r, from it will therefore be

$$-\frac{d}{dx}\left(\frac{\mu' A'}{r}\,dydz\right)dx = \mu' A' \frac{x}{r^3}\,dxdydz.$$

At every point of the medium the current is then made up of two parts, $4\pi\mu' A'$ and another which we may call $\mu'\alpha'$. So that we may write for the component of the current

$$a' = \mu'(\alpha' + 4\pi A').$$

We can deduce a similar equation for magnetism, indeed, one that is already known, except that we shall have what I call the magneto-motive force (similar to coercive force as used by some) in the place of the magnetization. We may then write out the equations as follows:

$$a = \mu(\alpha + 4\pi A) \qquad\qquad a' = \mu'(\alpha' + 4\pi A')$$

$$b = \mu(\beta + 4\pi B) \qquad\qquad b' = \mu'(\beta' + 4\pi B')$$

$$c = \mu(\gamma + 4\pi C). \qquad\qquad c' = \mu'(\gamma' + 4\pi C').$$

The quantities α, β, and γ are evidently the components of the magnetic *force* in the direction of the axes, and a, b, c of the magnetic induction.

The quantities α', β', and γ' are evidently the components of the electric force in the direction of the axes, and a', b', c' of the current.

The quantities a, b, c, a', b', c', and α, β, γ, α', β', γ' differ essentially from each other, inasmuch as the first indicate cycles, but the second are necessarily acyclic. The quantities indicated by the Greek letters can always be obtained from acyclic potentials.

In this case we have, taking one equation as an illustration,

$$\alpha = \frac{dV}{dx}$$

$$\mu\alpha = -\Delta^2 L_1 + \frac{dJ}{dx},$$

$$\text{or}\quad \Delta^2 L_1 = \frac{d}{dx}(J - \mu V).$$

Hence, by changing the values of χ and χ' from their former values to

$$\chi = \frac{1}{\mu}\iiint (J - \mu V)\frac{1}{r}\,dxdydz, \qquad\qquad \chi' = \frac{1}{\mu'}\iiint (J' - \mu' V')\frac{1}{r}\,dxdydz,$$

we can write

$$L = \mu \left\{ \iiint \frac{A}{r}\, dxdydz - \frac{d\chi}{dx} \right\} \qquad L' = \mu' \left\{ \iiint \frac{A'}{r}\, dxdydz - \frac{d\chi'}{dx} \right\}$$

$$M = \mu \left\{ \iiint \frac{B}{r}\, dxdydz - \frac{d\chi}{dy} \right\} \qquad M' = \mu' \left\{ \iiint \frac{B'}{r}\, dxdydz - \frac{d\chi'}{dy} \right\}$$

$$N = \mu \left\{ \iiint \frac{C}{r}\, dxdydz - \frac{d\chi}{dz} \right\} \qquad N' = \mu' \left\{ \iiint \frac{C'}{r}\, dxdydz - \frac{d\chi'}{dz} \right\}$$

$$F = \frac{\mu}{4\pi} \iiint \left\{ p\left(\frac{dB}{dx} - \frac{dA}{dy}\right) + B\frac{dp}{dx} - A\frac{dp}{dy} \right\} dxdydz \qquad F' = \frac{\mu'}{4\pi} \iiint \left\{ p\left(\frac{dB'}{dx} - \frac{dA'}{dy}\right) + B'\frac{dp}{dx} - A'\frac{dp}{dy} \right\} dxdydz$$

$$G = \frac{\mu}{4\pi} \iiint \left\{ p\left(\frac{dA}{dz} - \frac{dC}{dx}\right) + A\frac{dp}{dz} - C\frac{dp}{dx} \right\} dxdydz \qquad G' = \frac{\mu'}{4\pi} \iiint \left\{ p\left(\frac{dA'}{dz} - \frac{dC'}{dx}\right) + A'\frac{dp}{dz} - C'\frac{dp}{dx} \right\} dxdydz$$

$$H = \frac{\mu}{4\pi} \iiint \left\{ p\left(\frac{dB}{dx} - \frac{dA}{dy}\right) + B\frac{dp}{dx} - A\frac{dp}{dy} \right\} dxdydz. \qquad H' = \frac{\mu'}{4\pi} \iiint \left\{ p\left(\frac{dB'}{dx} - \frac{dA'}{dy}\right) + B'\frac{dp}{dx} - A'\frac{dp}{dy} \right\} dxdydz.$$

These equations have thus been put into a form depending only on the electro-motive and magneto-motive forces, an extremely important modification. All the equations which we have so far obtained are entirely independent of any relation between electricity and magnetism; and they are perfectly true, whether that relation is the one hitherto known or also includes the relation just discovered. The equations on the left hand are all more or less known, but the similar equations on the right are for the most part new. I shall first give the relations between the two systems of equations on the old theory, so as to contrast them with the new. We remark, in the first place, that it is supposed in these equations that the electric currents are due entirely to electro-motive forces, and the magnetism to magneto-motive forces. But where closed electric currents exist, magnetism must always exist; and thus to calculate the magnetic effect by these equations, we must replace each electric current by its equivalent magneto-motive shell. It is readily seen that this will be accomplished by the addition of terms depending on the current, and hence we may write

$$\frac{dc}{dy} - \frac{db}{dz} = 4\pi\mu \left\{ a' + \frac{dC}{dy} - \frac{dB}{dz} \right\}$$

$$\frac{da}{dz} - \frac{dc}{dx} = 4\pi\mu \left\{ b' + \frac{dA}{dz} - \frac{dC}{dx} \right\}$$

$$\frac{db}{dx} - \frac{da}{dy} = 4\pi\mu \left\{ c' + \frac{dB}{dx} - \frac{dA}{dy} \right\}.$$

Equations similar to this were first given by Sir William Thomson, except

that his applied to magnetic force rather than magnetic induction, and so did not contain the terms in A, B, and C. They would thus be

$$\frac{d\gamma}{dy} - \frac{d\beta}{dz} = 4\pi a'$$

$$\frac{d\alpha}{dz} - \frac{d\gamma}{dx} = 4\pi b'$$

$$\frac{d\beta}{dx} - \frac{d\alpha}{dy} = 4\pi c'.$$

As equations of this form exactly define α, β, and γ, in terms of A', B', and C', we must have

$$4\pi F' = \alpha, \qquad 4\pi G' = \beta, \qquad 4\pi H' = \gamma$$

and

$$4\pi\mu\left(L' + \mu'\frac{d\chi'}{dx}\right) = F - \mu\iiint\left\{\left(\frac{d(pC)}{dy} - \frac{d(pB)}{dz}\right)\right\}dxdydz$$

$$4\pi\mu\left(M' + \mu'\frac{d\chi'}{dy}\right) = G - \mu\iiint\left\{p\left(\frac{d(pA)}{dz} - \frac{d(pC)}{dx}\right)dxdydz\right.$$

$$4\pi\mu\left(N' + \mu'\frac{d\chi'}{dz}\right) = H - \mu\iiint\left\{p\left(\frac{d(pB)}{dx} - \frac{d(pA)}{dy}\right)dxdydz.\right.$$

Where there is no magneto-motive force (no permanent magnets), and we choose L', M', and N', so as to satisfy the equation of continuity, we have

$$4\pi\mu L' = F, \qquad 4\pi\mu' M' = G, \qquad 4\pi\mu' N' = H.$$

From the almost perfect reciprocity between electric currents and magnetic induction which I have developed in this paper, we might be led to expect that similar expressions might be found for the other case. But this is not so according to our usually conceived ideas, for lines of magnetic force must always exist around a wire carrying a current, but currents need not necessarily exist around magnets.

As we know that all the electric currents and all the magnetic force must depend only upon the terms containing A', B', and C', and A, B, and C, therefore we may select a series of terms as follows to represent the final result:

$$L = \mu\left\{\iiint Apdxdydz - \frac{d\chi}{dx}\right\}$$

$$M = \mu\left\{\iiint Bpdxdydz - \frac{d\chi}{dy}\right\}$$

$$N = \mu\left\{\iiint Cpdxdydz - \frac{d\chi}{dz}\right\}$$

$$F = \frac{dN}{dy} - \frac{dM}{dz} + 4\pi\mu\mu' \left\{ \iiint A'pdxdydz - \frac{d\chi'}{dx} \right\}$$

$$G = \frac{dL}{dz} - \frac{dN}{dx} + 4\pi\mu\mu' \left\{ \iiint B'pdxdydz - \frac{d\chi'}{dy} \right\}$$

$$H = \frac{dM}{dx} - \frac{dL}{dy} + 4\pi\mu\mu' \left\{ \iiint C'pdxdydz - \frac{d\chi'}{dz} \right\}.$$

$$a = \frac{dH}{dy} - \frac{dG}{dz} \qquad 4\pi\mu a' = \frac{dc}{dy} - \frac{db}{dz} - 4\pi\mu \left\{ \frac{dC}{dy} - \frac{dB}{dz} \right\}$$

$$b = \frac{dF}{dz} - \frac{dH}{dx} \qquad 4\pi\mu b' = \frac{da}{dz} - \frac{dc}{dx} - 4\pi\mu \left\{ \frac{dA}{dz} - \frac{dC}{dx} \right\}$$

$$c = \frac{dG}{dz} - \frac{dF}{dy} \qquad 4\pi\mu c' = \frac{db}{dx} - \frac{da}{dy} - 4\pi\mu \left\{ \frac{dB}{dx} - \frac{dA}{dy} \right\}.$$

We know that every magnetic action can be represented by electric currents, therefore let us suppose the permanent magnets to be replaced by their equivalent electric currents. We then have simply

$$F = 4\pi\mu\mu' \left\{ \iiint A'pdxdydz - \frac{d\chi'}{dx} \right\}$$

$$G = 4\pi\mu\mu' \left\{ \iiint B'pdxdydz - \frac{d\chi'}{dy} \right\}$$

$$H = 4\pi\mu\mu' \left\{ \iiint C'pdxdydz - \frac{d\chi'}{dz} \right\}$$

$$a = \frac{dH}{dy} - \frac{dG}{dz} \qquad 4\pi\mu a' = \frac{dc}{dy} - \frac{db}{dz}$$

$$b = \frac{dF}{dz} - \frac{dH}{dx} \qquad 4\pi\mu b' = \frac{da}{dz} - \frac{dc}{dx}$$

$$c = \frac{dG}{dx} - \frac{dF}{dy} \qquad 4\pi\mu c' = \frac{db}{dx} - \frac{da}{dy}.$$

These equations at first sight seem to be similar to those already in use, and which are given in Maxwell's "Treatise," Art. 616. But if we examine them further, we shall see that the values of F, G, and H are given in terms of A', B', and C', rather than in terms of a', b', and c', and the introduction of these electro-motive forces together with the idea of magneto-motive forces, are the principal new points in the above theory, and are very necessary to the further extension of the theory to include the new electro-magnetic action.

III. *Physical Theory of Magnetic Attraction and Repulsion.*

The form in which the formulæ are thus thrown by expressing them in terms of the electro-motive forces has suggested to me the following physical theory of magnetism, which I give here before proceeding further.

They evidently point to the existence of a fluid filling all space, the components of whose velocity are $-F$, $-G$, and $-H$, the vortex filaments of which are the lines of magnetic induction and the vortices of the vortices, or the relative motions, of which constitute the electric currents. The motion of this fluid, we see, depends upon quantities of two kinds. The first kind includes the terms containing A', B', and C', or the electro-motive forces; and these forces must be conceived existing at various points, and tending to propel and rotate the fluid in certain directions. The second series of terms containing χ' are those portions of the components of the fluid motion which it is possible to produce by the existence in the fluid of points at which the fluid is being constantly generated or destroyed, or by moving bodies within the fluid. In other words, they indicate the motion of translation of the fluid. The calculation of χ' is thus necessary to make F, G, and H satisfy the equation of continuity, and thus to represent the components of fluid motion; but as they disappear from the succeeding equations, they are not necessary for the calculation of either the magnetic action or the electric currents.

The disappearance of χ' from the succeeding equations gives the theorem in vortex motion that no vortices can be produced in a fluid by external forces, and in electricity the theorem, which I have before demonstrated, *that electric currents in a continuous medium can never produce magnetic action unless they are closed.*

We have thus to conceive of a medium, the ordinary motions of which do not produce magnetic action or constitute electric currents, but the vortices of which are the magnetic lines of force, and the *relative* motions the electric currents.

The idea of such an extended conducting medium with its electric currents has thus lead us to the idea of a fluid filling all space, of whose ordinary motions we are unconscious, and which the earth may be whirling through with its full velocity without our being conscious of it, seeing that no magnetic action would be thus produced. But does not this give a possible explanation of the magnetism of the earth, seeing that the earth's rotation might produce rotation of this fluid which would be magnetic action? We might call this fluid *ether* if we pleased; but I do not wish this undeveloped theory to be confounded with the very crude, unscientific, and altogether untenable theory of Edlund; for

the fluid which I conceive of has none of the properties of Edlund's so-called ether.

Whether this theory could be adapted to explain electrostatic phenomena, and hence the propagation of light on Maxwell's theory, I have not yet determined. But it is to be noted that Maxwell's equations are in terms of F, G, and H, and indicate that the waves of light are waves in the fluid of which I conceive. *Could a property be added to this fluid to explain electrostatic phenomena, the fluid would therefore be identical in properties with the light ether.* This property is that on which the so-called electric displacement of Maxwell depends. Maxwell has worked up a theory similar to this in some respects, in the Phil. Mag. for 1861 and 1862; which, however, required a very artificial constitution of the ether, as he could not conceive of any method of making vortex rings in a perfect fluid. But I will show further on that my idea of electro-motive force gets over this difficulty, and allows us to explain all magnetic attractions and repulsions by the motion of a *perfect* fluid.

To get the true motion of the fluid we must be able to compute χ', although this is unnecessary as far as the magnetic action or the electric currents are concerned. To accomplish this, it is best to investigate the value of χ' for an element of electro-motive force, and then it will always appear in other cases as a definite integral. The values for an element of electro-motive force, $A'dxdydz$, at the origin are

$$F = 4\pi\mu\mu' \left\{ \frac{A'}{r}\, dxdydz - \frac{d\chi'}{dx} \right\}$$

$$G = -\, 4\pi\mu\mu' \,\frac{d\chi'}{dy}$$

$$H = -\, 4\pi\mu\mu' \,\frac{d\chi'}{dz}.$$

The equation of continuity gives therefore

$$4\pi\mu\mu' \left\{ A' \frac{x - x'}{r^3}\, dxdydz + \Delta^2\chi' \right\} = 0,$$

whence
$$\chi' = \frac{A'dxdydz}{4\pi} \iiint \frac{x - x'}{r^3 R}\, dxdydz.$$

But as this integration is very difficult, I make use of the theorem

$$\Delta^2 r = \frac{2}{r},$$

which gives by differentiation

$$\Delta^2\left(\frac{x-x'}{r}\right) = -2\,\frac{x-x'}{r^3},$$

and hence
$$\chi' = \frac{A'}{2}\,\frac{x-x'}{r}\,dxdydz.$$

If we also had at the given point electro-motive forces B' and C', then we should have finally

$$F = 4\pi\mu\mu'\left\{\frac{A'}{r}\,dxdydz - \frac{d\chi'}{dx}\right\}$$

$$G = 4\pi\mu\mu'\left\{\frac{B'}{r}\,dxdydz - \frac{d\chi'}{dy}\right\}$$

$$H = 4\pi\mu\mu'\left\{\frac{C'}{r}\,dxdydz - \frac{d\chi'}{dz}\right\},$$

where in general

$$\chi' = \frac{1}{2r}\left\{A'(x-x') + B'(y-y') + C'(z-z')\right\}dxdydz.$$

χ' will evidently disappear when A', B', and C' satisfy the equation of continuity. This happens when the electro-motive force forms circuits which are perfectly closed.

The integrals of these expressions taken throughout space will give the values for any case.

As the motion of the imaginary fluid is rotational, it is at first sight very difficult to conceive how this rotation can be produced. For Helmholtz has shown that vortex filaments can neither be created nor destroyed by any forces acting on the fluid from without. Our problem is to conceive how an electro-motive force of intensity A', acting within an element, can fill the whole space with vortex rings, whose intensities are given by the equations.

At first I was puzzled, but finally conceived the following solution. Let the nature of electro-motive force be such that it tends to form vortex rings immediately around itself, not by action at a distance, but by direct action on the fluid in the immediate vicinity. The first ring will move forward, another one will form, and so on until all space is filled with them, when there will be equilibrium. I have not yet attempted the whole dynamics of the subject.

Magnetic attractions can be explained as follows. Conceive the fluid in a tube to be rotating around its axis with a certain velocity, and suppose the ends of the tube to be closed with movable pistons. Then, if the pistons are left

free, there will be a centrifugal force against the sides of the tube proportional to the square of the velocity of angular rotation. If the walls are flexible and the pistons immovable, then there will be a force tending to press the pistons in, and proportional also to the square of the velocity.

According to our theory the magnetic force is the velocity of rotation, and so we have in the medium a tension along the lines of force, and a pressure at right angles to them.

To satisfy the conservation of energy and the equation of continuity of the fluid, the work done on the pistons or on the envelope, together with the volume passed over, must be the same; and hence the pressure and the tension must be the same numerically.

Now Maxwell has shown that all magnetic attractions or repulsions can be accounted for by a tension along the lines of magnetic force, together with an equal pressure at right angles to them. *Hence the motion of such a fluid as we have been considering will account for all magnetic action either of magnets or of electric currents.*

Again, Sir William Thomson has shown, in his analogy of magnetism to fluid motion, that two points in a liquid at which the liquid is generated, so that there are constant currents out from them, *attract* each other. In other words, his analogy is not perfect; for, although the lines of force are of the same shape as the stream lines, yet like poles would attract, and unlike repel. But the theory of this paper, which considers lines of force as vortex filaments in a *perfect* fluid, gives a true analogy, which we have seen above accounts for all magnetic action. But Sir William Thomson's principle does not explain such action, for it would vanish for a closed circuit; and would thus give that term which has an arbitrary value in Ampère's theory, and which likewise vanishes for a closed circuit.

All the motion of translation of the fluid represented by the terms χ' disappears when the electro-motive force forms closed circuits; that is, when the components of the electro-motive force satisfy the equation of continuity throughout space. The energy of the fluid will then be merely that of the vortex filaments all through it; or, in other words, the energy will be entirely magnetic. It is evident without calculation, therefore, that this kind of fluid explains the magnetic attraction of currents without further hypothesis, and that the equation of Ampère, or some other giving the same results for closed circuits, could be readily deduced.

The principal advantage of this theory over that of Maxwell is in the idea of electro-motive force here introduced, by which an ordinary perfect fluid becomes available instead of one having a complicated and unknown law

connecting its molecules. But I do not yet attempt to explain electrostatic action.

The idea of electro-motive force here introduced has thus rotation as its basis. And by this idea we have thus done away with the necessity of any action similar to viscosity in the medium, but have shown that a force producing vortices at any one point will fill all space with vortices, which will be packed together according to the laws indicated by the equations.

I do not consider this theory as final, by any means, but only as one link in the chain, the first three links of which have been added by Thomson, Helmholtz, and Maxwell, and which *may* finally end in the true theory.

IV. *Extension of Equations to include the newly discovered Electro-magnetic Action.*

Leaving this theory, however tempting further discussion might be, and considering the equations again merely from an electrical point of view, we have now to inquire how the recently discovered electro-magnetic action will affect them. The following theory may be regarded as only preliminary.

Evidently we must add to the electro-motive forces which produce the original currents others depending on the current and the magnetic force at each point. These new electro-motive forces have been found to be linear functions of the magnetic force and the current at each point. At present they appear to be at right angles to the plane containing these two, but for the sake of generality I will first suppose the function to be general. Let the new electro-motive forces be A'', B'', and C'', and let $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, and $\mathfrak{d}$ be four new constants depending on the material. Then we can write evidently

$$
\begin{aligned}
A'' &= \mathfrak{a}a\sqrt{b'^2+c'^2} + \mathfrak{b}a'\sqrt{b^2+c^2} + \mathfrak{c}\,(bc'-b'c) + \mathfrak{d}aa' \\
B'' &= \mathfrak{a}b\sqrt{a'^2+c'^2} + \mathfrak{b}b'\sqrt{a^2+c^2} + \mathfrak{c}\,(ca'-c'a) + \mathfrak{d}bb' \\
C'' &= \mathfrak{a}c\sqrt{a'^2+b'^2} + \mathfrak{b}c'\sqrt{a^2+b^2} + \mathfrak{c}\,(ab'-a'b) + \mathfrak{d}cc'.
\end{aligned}
$$

Of these four constants, Mr. Hall has proved the existence of $\mathfrak{c}$, and also that $\mathfrak{a}$ is very small or zero. As to the other two constants, it would seem almost an impossibility to test their value with accuracy; Mr. Hall has attempted to do so by trying the resistance of a wire and also of a piece of gold leaf under magnetic action, but has as yet determined only that the effect is almost inappreciable on the resistance.

But if we suppose the action to be rotatory, as I pointed out in the last number of this journal, we can get a probable relation between $\mathfrak{c}$ and $\mathfrak{b}$. For let

$b' = c = a = a' = 0$, and we have the case of a current, c', and a magnetic force, b, and the equations reduce to

$$A'' = \mathfrak{c}bc'$$
$$B'' = \mathfrak{a}bc' = 0$$
$$C'' = \mathfrak{b}bc'.$$

If the action is a rotation of the current element around the line of force, we should have this rotation proportional to b and $\mathfrak{b}$ would be to $\mathfrak{c}$ in the ratio of the tangent of the angle of rotation to unity. But as the angle must be very minute, $\mathfrak{b}$ must be also very small compared with unity.

I have attempted to apply the conservation of energy to the case, but have not yet obtained any very good results, especially as conduction in a moving medium does not seem to be perfectly understood.

Let us then suppose all the constants except $\mathfrak{c}$ to be zero, and we then have

$$A'' = \mathfrak{c}\,(bc' - b'c)$$
$$B'' = \mathfrak{c}\,(ca' - c'a)$$
$$C'' = \mathfrak{c}\,(ab' - a'b).$$

The general equations for the electro-magnetic action alone will then be

$$F = 4\pi\mu\mu' \left\{ \iiint \left[A' + \mathfrak{c}\,(b_1c' - b'c_1) \right] \frac{1}{r}\, dxdydz - \frac{d\chi'}{dx} \right\}$$

$$G = 4\pi\mu\mu' \left\{ \iiint \left[B' + \mathfrak{c}\,(c_1a' - c'a_1) \right] \frac{1}{r}\, dxdydz - \frac{d\chi'}{dy} \right\}$$

$$H = 4\pi\mu\mu' \left\{ \iiint \left[C' + \mathfrak{c}\,(a_1b' - a'b_1) \right] \frac{1}{r}\, dxdydz - \frac{d\chi'}{dz} \right\},$$

where a_1, b_1, and c_1 indicate the *total* components of the magnetism due to magnets as well as electric currents, and a, b, and c are the components due to the currents alone.

F, G, and H do not include the vector potentials of the magnets.

$$a = \frac{dH}{dy} - \frac{dG}{dz} \qquad\qquad 4\pi\mu a' = \frac{dc}{dy} - \frac{db}{dz}$$

$$b = \frac{dF}{dz} - \frac{dH}{dx} \qquad\qquad 4\pi\mu b' = \frac{da}{dz} - \frac{dc}{dx}$$

$$c = \frac{dG}{dx} - \frac{dF}{dy} \qquad\qquad 4\pi\mu c' = \frac{db}{dx} - \frac{da}{dy}.$$

If we wish F, G, and H to satisfy the equation of continuity, then the value of χ' will be, as we have found before,

$$\chi' = \frac{1}{2}\iiint \frac{1}{r}\left\{[A'+\mathfrak{C}(b_1c'-b'c_1)][x-x']+[B'+\mathfrak{C}(c_1a'-c'a_1)][y-y']+[C'+\mathfrak{C}(a_1b'-a'b_1)][z-z']\right\}dxdydz.$$

Otherwise we can give it any value we please, as it vanishes from the other equations. In case we make it zero, we shall have *

$$4\pi\mu a' = -\Delta^2 F$$

$$4\pi\mu b' = -\Delta^2 G$$

$$4\pi\mu c' = -\Delta^2 H.$$

The exact calculation will in general be very complicated. But as $\mathfrak{C}$ is very small in all substances so far experimented on, we can easily calculate the effect as a correction to the quantities calculated on the ordinary theory. But where the effect is to be calculated in a limited body, it then, in general, becomes very complicated. However, the solution is possible by ordinary integration for a thin circular disc or for a sphere, as we can then apply the method of images.

In some cases no currents will be produced in the body, but simply a difference of potential which can be measured by a galvanometer, as in Mr. Hall's experiment.

V. *Explanation of the Magnetic Rotation of the Plane of Polarization of Light.*

To apply these results to Maxwell's theory of light, we must assume that the same action which takes place in conductors with reference to conducted currents, also takes place in dielectrics with reference to displacement currents. It is almost impossible to detect this action experimentally, but we shall here follow out the consequence of its existence. I shall follow the method of Art. 783 of Maxwell's "Treatise," with the addition of this new action.

Assume at once $C=0$, $\psi=0$, and $J=0$ as they are afterwards taken or proved to be.

Let P, Q, and R be the components of the electro-motive forces acting at any point. The electro-motive force will be composed of two parts: first, the rate of variation of the vector potential as on the old theory; and, second, a term

* I use the expression Δ^2 to signify the operation $\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}$, while Maxwell uses it in his *Theory of Light* with the opposite sign.

depending on the new action, and whose components we have designated by A', B', and C'. Adding these together, we have

$$P = -\frac{dF}{dt} - C(b_1c' - b'c_1)$$

$$Q = -\frac{dG}{dt} - C(c_1a' - c'a_1)$$

$$R = -\frac{dH}{dt} - C(a_1b' - a'b_1).$$

The displacement currents a', b', and c' will be

$$a' = \frac{K}{4\pi}\frac{dP}{dt}$$

$$b' = \frac{K}{4\pi}\frac{dQ}{dt}$$

$$c' = \frac{K}{4\pi}\frac{dR}{dt},$$

and they are also expressed by the equations *

$$4\pi\mu a' = -\Delta^2 F$$

$$4\pi\mu b' = -\Delta^2 G$$

$$4\pi\mu c' = -\Delta^2 H.$$

Hence we have by elimination

$$K\mu\left\{\frac{d^2F}{dt^2} - \frac{d}{dt}C(b_1c' - b'c_1)\right\} - \Delta^2 F = 0$$

$$K\mu\left\{\frac{d^2G}{dt^2} - \frac{d}{dt}C(c_1a' - c'a_1)\right\} - \Delta^2 G = 0$$

$$K\mu\left\{\frac{d^2H}{dt^2} - \frac{d}{dt}C(a_1b' - a'b_1)\right\} - \Delta^2 H = 0.$$

Before the solution of these equations, of course the values of a_1, b_1, c_1 and a', b', c' must be substituted in terms of F, G, and H.

Let us now take the case of a plane polarized ray passing in the direction of the axis of z, with a magnetic force, c_1, along the same axis. The magnetic forces a, b, c, the variations of which constitute the waves of light, are very small; for Maxwell has calculated that in strong sunlight the maximum is about one-tenth of the horizontal intensity of the earth's magnetism. Hence we can write

$$K\mu\left\{\frac{d^2F}{dt^2}+\mathfrak{c}\frac{d}{dt}(b'c_1)\right\}-\frac{d^2F}{dz^2}=0$$

$$K\mu\left\{\frac{d^2G}{dt^2}-\mathfrak{c}\frac{d}{dt}(a'c_1)\right\}-\frac{d^2G}{dz^2}=0$$

$$K\mu\,\frac{d^2H}{dt^2}=0\,;$$

and, replacing b' and a' by their values, we have

$$K\mu\left\{\frac{d^2F}{dt^2}-\frac{\mathfrak{c}c_1}{4\pi\mu}\frac{d^3G}{dtdz^2}\right\}-\frac{d^2F}{dz^2}=0$$

$$K\mu\left\{\frac{d^2G}{dt^2}+\frac{\mathfrak{c}c_1}{4\pi\mu}\frac{d^3F}{dtdz^2}\right\}-\frac{d^2G}{dz^2}=0.$$

From the form of the equations we can well suppose that one solution is

$$F=r\cos(nt-qz)\cos mt$$

$$G=r\cos(nt-qz)\sin mt,$$

and making the substitution we find

$$\left\{K\mu(n^2+m^2)-q^2\left(1+\mathfrak{c}\frac{mc_1K}{4\pi}\right)\right\}\cos(nt-qz)\cos mt$$

$$-Kn\left\{2m\mu-\mathfrak{c}\frac{c_1q^2}{4\pi}\right\}\sin(nt-qz)\sin mt=0$$

$$\left\{K\mu(n^2+m^2)-q^2\left(1+\mathfrak{c}\frac{mc_1K}{4\pi}\right)\right\}\cos(nt-qz)\sin mt$$

$$+Kn\left\{2m\mu-\mathfrak{c}\frac{c_1q^2}{4\pi}\right\}\sin(nt-qz)\cos mt=0.$$

These are satisfied if we make the coefficients zero.

If V is the velocity in general of light in the medium, and V_0 the velocity in vacuo without magnetic action; if i is the index of refraction of the medium, and λ the complete wave length in the medium, and λ_0 in vacuo; we thus find

$$m=\mathfrak{c}\frac{\pi c_1}{2\mu\lambda^2}$$

$$V=\frac{n}{q}=\sqrt{\frac{1+\mathfrak{c}\frac{Kmc_1}{8\pi}}{K\mu}}$$

$$V=\frac{1}{\sqrt{K\mu}}\left(1+\frac{K\mu m^2\lambda^2}{8\pi^2}\right).$$

These equations indicate that when a ray of plane polarized light passes in the direction of the lines of magnetic force, the plane of polarization will be rotated in a direction depending on the sign of the quantity $\mathfrak{C}$, which is the well-known action of Faraday. But the second expression which gives the velocity, and consequently the index of refraction, also depends on $\mathfrak{C}$, and thus indicates an acceleration of the velocity which is unknown. But this action is so very minute that it can probably never be measured.

If D is the length of the substance, the total angle of rotation of the beam will evidently be

$$\theta = m\frac{D}{V} = \mathfrak{C}\frac{\pi}{2\mu V_0}\frac{i^3}{\lambda_0^2}Dc_1 .$$

This solution is rigorously exact for all cases where the index of refraction is not a function of the wave length. To get the value where the index varies, we can use the principle of the superposition of small quantities. Every plane polarized ray can be supposed to be made up of two circularly polarized rays; and to say that the plane of polarization is rotated simply means that one of the circularly polarized rays travels faster than the other; when one ray gains λ on the other the plane of polarization is rotated through the angle 2π. Hence, if V is the velocity of one and V' of the other, we have

$$V = V'\left(1 + \frac{\lambda}{D'}\right),$$

where D' is the distance in which the plane of polarization is rotated through the angle 2π.

But this effect will be augmented by the dispersion of the body, seeing that the velocity affects the wave length, and hence the index of refraction will be different for the two components. This further action can be taken into account by multiplying $\frac{V}{V'}$ by $\frac{i'}{i}$, and we then have

$$\frac{V}{V'}\frac{i'}{i} = 1 + \frac{\lambda}{D''}.$$

So that D' has been changed to

$$D'' = \frac{\lambda}{\frac{i'}{i}\left(1 + \frac{\lambda}{D'}\right) - 1}.$$

This can be put into the form

$$\frac{D''}{D'} = \frac{1}{\frac{i'}{i}\left\{1 + \frac{D'}{\lambda}\frac{i' - i}{i'}\right\}}.$$

But
$$\frac{\lambda}{\lambda'} = \frac{D' + \lambda}{D'},$$

whence
$$\lambda - \lambda' = \frac{\lambda^2}{D'} \qquad \text{and} \qquad \frac{i' - i}{i} = \frac{\lambda' - \lambda}{i} \frac{di}{d\lambda} + \text{etc.}$$

Hence, omitting all quantities of the second order of smallness, we can write

$$\frac{D''}{D'} = \frac{1}{1 - \frac{\lambda}{i} \frac{di}{d\lambda}};$$

and the angle of rotation, θ, will become

$$\theta = 2\pi \frac{D}{D'} = c D c_1 \frac{\pi}{2\mu V_0} \frac{i^2}{\lambda_0^2} \left(i - \lambda \frac{di}{d\lambda}\right),$$

which is of the same form as Maxwell's expression. Now Maxwell's equation is obtained from considerations entirely different from any which I have used in this paper. In obtaining them, Maxwell made no assumption as to the kind of motion which constitutes light, but merely assumed that the magnetic lines of force were vortices, and that the motion of the vortices caused a rotation of the motion constituting light. In my theory I have used no hypothesis as to the nature of magnetic force, but have simply calculated, from the known laws of magnetism and electricity, the action in this case according to Maxwell's theory of light. And the conclusion which we draw is that *the effect discovered by Mr. Hall is the same, or due to the same cause, as the rotation of the plane of polarization of light.*

It is interesting to repeat here the comparison made by Verdet between the various formulæ and observation.

The formulæ of Maxwell and Rowland, of Airy, and of Neumann are

$$\text{(I)} \qquad \theta = M \frac{i^2}{\lambda^2} \left(i - \lambda \frac{di}{d\lambda}\right) D c_1$$

$$\text{(II)} \qquad \theta = M \frac{1}{\lambda^2} \left(i - \lambda \frac{di}{d\lambda}\right) D c_1$$

$$\text{(III)} \qquad \theta = M \left(i - \lambda \frac{di}{d\lambda}\right) D c_1.$$

The comparison of these formulæ with the experiments of Verdet* are as follows:

* Verdet, *Œuvres*, Vol. I. p. 262.

Bisulphide of Carbon.

		C	D	E	F	G
Observed rotation,		0.592	0.768	1.000	1.234	1.704
Calculated, formula	I,	0.589	0.760	1.000	1.234	1.713
" "	II,	0.606	0.772	1.000	1.216	1.640
" "	III,	0.943	0.967	1.000	1.034	1.091

Creosote.

		C	D	E	F	G
Observed rotation,		0.573	0.758	1.000	1.241	1.723
Calculated, formula	I,	0.617	0.780	1.000	1.210	1.603
" "	II,	0.627	0.789	1.000	1.200	1.565
" "	III,	0.976	0.993	1.000	1.017	1.041

To examine the direction of the action, we must see what the relative direction of the currents and magnetism are in the equations, as I have not taken the signs with respect to any system.

Let the positive direction of the current be the direction in which the positive electricity moves, and the positive direction of the magnetic lines of force be the direction in which the north pole tends to move. Then we easily find that our equations are on the right-handed screw system, the right-handed screw being such that if we turn it in the direction of the hands of a watch with its face toward us, it will move away from us. According to this system, Mr. Hall has found that the value of $\mathfrak{C}$ is positive for gold and some other diamagnetic substances, and negative for iron. Hence a magnetic force in the positive direction will cause the ray to be rotated in the positive direction in diamagnetic substances, and in the negative direction in magnetic ones, which is exactly what has been observed.

To compare the numerical amount of the revolution with observation, we can take the constants as observed by Mr. Hall for gold, and thus find at least whether it is of the proper order of magnitude.

From more recent observations than those published, Mr. Hall finds that, in the field of his magnet, he can cause the lateral electro-motive force to be at least as great as $\frac{1}{2000}$ of the force along the strip. According to the system of units used in this paper, the new electro-motive force will be in the case of conduction, the current passing along V and the magnetism being in the direction of z,

$$A'' = -\,\mathfrak{C}c_1 b' = -\,4\pi\mu'\mathfrak{C}c_1 B';$$

but Mr. Hall finds

$$\frac{B'}{A''} = -\,2000 \text{ nearly.}$$

Hence, using the C. G. S. system, in which $\mu' = 2000$ nearly, we shall have

$$cc_1 = \frac{1}{4\pi} \text{ nearly for gold.}$$

The length of the substance in which the ray is rotated a complete revolution, or 360°, will then be

$$D = \frac{2\pi V_0}{mi} = \frac{4\,\mu\lambda_0^2\, V_0}{cc_1'^2 \left(i - \lambda \frac{di}{d\lambda}\right)},$$

where it is to be noted that λ_0 is the length of a *complete* wave. Taking the wave of $\frac{1}{10000}$ cm. length, and the index of refraction 4, we find, supposing $\frac{di}{d\lambda} = 0$,

$$D = 240 \text{ cm. nearly.}$$

We do not know the magnetic force used by Verdet, but it was evidently of the same order of magnitude. He found D to be about as follows: 300 for heavy glass, 700 for flint glass. Hence the rotation calculated for gold is of the same order of magnitude as the rotation observed in some common substances.

Thus the new electro-magnetic phenomenon explains in the most perfect manner the magnetic rotation of the plane of polarization of light, and we are almost in the position to pronounce positively that the two phenomena are the same. Should this preliminary theory of the subject stand the test of time, it hardly seems to me that we can regard it in any other light than a demonstration of the truth of Maxwell's theory of light; for the rotation of the plane of polarization is thus a necessary consequence of the laws of electro-magnetism, and this, added to the other facts of the case, raises Maxwell's theory almost to the realm of fact.

Orthomorphic Projection of an Ellipsoid upon a Sphere.

BY THOMAS CRAIG,

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AFTER the labors of such men as Gauss, Lagrange, and Lambert, upon the general theory of orthomorphic projection, or projection by similarity of infinitesimal areas, it does not seem as though much of value or interest could be obtained by any further study of the subject, and indeed there is nothing more to be said upon the general theory, but an abundance of opportunities still exists for applying the results obtained by these princes of the realm of mathematics to the solution of particular cases of the problem. Boole, in the supplementary volume to his Differential Equations (a posthumous work, edited by Todhunter), gives a most elegant investigation of orthomorphic projection upon a plane; the formulæ arrived at are applicable to any surface which it may be desired to project upon a plane. One application is made by Boole to the case of an oblate spheroid such as the earth. I have given in another place an application of his formulæ to the projection of the general ellipsoid upon a plane, and the results obtained there led me to attempt the projection of the ellipsoid upon a sphere. Gauss gives, among the examples illustrative of his general theory, the projection of an ellipsoid of revolution upon a sphere, and considers the so-solved problem to be a valuable addition to geodesy. At the present day, when it is at least considered possible that the earth may be a general ellipsoid, it may be that the formulæ necessary for its projection upon a sphere will not be devoid of interest.

Denote by R the radius of the sphere, then its equation may be given as

$$\xi^2 + \eta^2 + \zeta^2 = R^2; \tag{1}$$

if U and V are the usual spherical co-ordinates, we have for ξ, η, ζ, the values

$$\begin{aligned} \xi &= R \cos U \sin V, \\ \eta &= R \sin U \sin V, \\ \zeta &= R \cos V. \end{aligned} \tag{2}$$

Denote by a, b, c, the semi-axes of the ellipsoid, then this surface is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\,; \tag{3}$$

if λ_1 and λ_2 denote the variable parameters belonging to the two hyperboloids confocal to the given ellipsoid, we have for these surfaces the equations

$$\begin{aligned}\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}+\frac{z^2}{c^2+\lambda_1}=1,\\ \frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}+\frac{z^2}{c^2+\lambda_2}=1,\end{aligned} \tag{4}$$

and then, as is well known, the co-ordinates x, y, z, are given by

$$\begin{aligned} x^2&=\frac{a^2\,(a^2+\lambda_1)\,(a^2+\lambda_2)}{(a^2-b^2)\,(a^2-c^2)}\\ y^2&=\frac{b^2\,(b^2+\lambda_1)\,(b^2+\lambda_2)}{(b^2-c^2)\,(b^2-a^2)}\\ z^2&=\frac{c^2\,(c^2+\lambda_1)\,(c^2+\lambda_2)}{(c^2-a^2)\,(c^2-b^2)}\end{aligned} \tag{5}$$

and the element of length on the ellipsoid by

$$\Omega=dS^2=\frac{(\lambda_1-\lambda_2)}{4}\left\{\frac{\lambda_1 d\lambda_1^2}{(a^2+\lambda_1)\,(b^2+\lambda_1)\,(c^2+\lambda_1)}+\frac{\lambda_2 d\lambda_2^2}{(a^2+\lambda_2)\,(b^2+\lambda_2)\,(c^2+\lambda_2)}\right\}. \tag{6}$$

Write for brevity

$$\begin{aligned} L_1&=\sqrt{(a^2+\lambda_1)\,(b^2+\lambda_1)\,(c^2+\lambda_1)},\\ L_2&=\sqrt{(a^2+\lambda_2)\,(b^2+\lambda_2)\,(c^2+\lambda_2)}\,;\end{aligned} \tag{7}$$

the Gaussian equation

$$\Omega=0$$

leads then to

$$\frac{\sqrt{\lambda_1}\,d\lambda_1}{L_1}\mp i\,\frac{\sqrt{\lambda_2}\,d\lambda_2}{L_2}=0, \tag{8}$$

the differential equations of the problem.

For the element of length on the sphere equations (2) give

$$\Omega^1=dS^2=R^2\sin^2 V dU^2+R^2dV^2. \tag{9}$$

$\Omega^1=0$ leads to the differential equations

$$dU\mp i\,\frac{dV}{\sin V}=0,$$

and from this follows by integration

$$U\pm i\,\log\cot\tfrac{1}{2}V=\text{const.} \tag{10}$$

If we find one integral of (8), in the form

$$P+iQ=\text{const.},$$

the problem will be solved by equating U to the real, and $i \log \cot \frac{1}{2} V$ to the imaginary part of

$$f(P + iQ);$$

f being an arbitrary functional symbol and $P + iQ$ being the argument of the function, all operations having to be performed upon this complex-quantity as a whole and not upon its components.

The first thing to be done is, of course, to so transform (8) that an integration may be performed. To this end, observe that the parameters λ_1 and λ_2 are limited by the relations

$$-c^2 > \lambda_1 > -b^2,$$
$$-b^2 > \lambda_2 > -a^2,$$

so that we can express λ_1 in terms of a new variable θ as

$$\lambda_1 = -\frac{b^2(a^2-c^2)\cos^2\theta + c^2(a^2-b^2)\sin^2\theta}{(a^2-c^2)\cos^2\theta + (a^2-b^2)\sin^2\theta}; \tag{11}$$

or again, writing

$$\frac{a^2-b^2}{a^2-c^2}\tan^2\theta = \omega^2, \tag{12}$$

$$\lambda_1 = -\frac{b^2+c^2\omega^2}{1+\omega^2}. \tag{13}$$

From these we have the relations

$$\left.\begin{aligned}
a^2+\lambda_1 &= \frac{(a^2-b^2)+(a^2-c^2)\omega^2}{1+\omega^2}, & P &= \int\frac{\sqrt{\lambda_1}\,d\lambda_1}{L_1},\\
b^2+\lambda_1 &= \frac{(b^2-c^2)\omega^2}{1+\omega^2}, & \lambda_1 &= -\frac{b^2+c^2\omega^2}{1+\omega^2},\\
c^2+\lambda_1 &= -\frac{(b^2-c^2)}{1+\omega^2}, & d\lambda_1 &= \frac{(b^2-c^2)\,2\,\omega d\omega}{(1+\omega^2)^2}.
\end{aligned}\right\} \tag{14}$$

These substitutions made in P give after simple reductions

$$P = \int\frac{\sqrt{b^2+c^2\omega^2}}{\sqrt{(a^2-b^2)+(a^2-c^2)\omega^2}}\,\frac{d\omega}{1+\omega^2}. \tag{15}$$

For the transformation back to the variable θ observe that we have

$$\left.\begin{aligned}
b^2+c^2\omega^2 &= \frac{b^2(a^2-c^2)-a^2(b^2-c^2)\sin^2\theta}{(a^2-c^2)\cos^2\theta},\\
d\omega &= \sqrt{\frac{a^2-b^2}{a^2-c^2}}\sec^2\theta\,d\theta,\\
1+\omega^2 &= \frac{(a^2-c^2)\cos^2\theta}{(a^2-c^2)-(b^2-c^2)\sin^2\theta},\\
\sqrt{(a^2-b^2)+(a^2-c^2)\omega^2} &= \frac{\sqrt{a^2-b^2}}{\cos\theta};
\end{aligned}\right\} \tag{16}$$

multiplication of the last three of these gives

$$\frac{2\sqrt{a^2-c^2}\cos\theta\, d\theta}{(a^2-c^2)-(b^2-c^2)\sin^2\theta}.$$

The result of substituting in P these values is

$$P=\sqrt{\frac{b^2}{a^2-c^2}}\int\frac{\sqrt{\left[1-\frac{a^2(b^2-c^2)}{b^2(a^2-c^2)}\sin^2\theta\right]}}{1-\frac{b^2-c^2}{a^2-c^2}\sin^2\theta}\,d\theta. \tag{17}$$

The quantity

$$\frac{b^2-c^2}{a^2-c^2}<1$$

occurs in both numerator and denominator of the differential expression; in the numerator occurs also the factor $\frac{a^2}{b^2}$; for $a^2=b^2$,

$$\frac{a^2}{b^2}\cdot\frac{b^2-c^2}{a^2-c^2}=1;$$

for $c^2=b^2$, this is $=0$; so that the value of

$$\frac{a^2}{b^2}\cdot\frac{b^2-c^2}{a^2-c^2}$$

lies always between the limits 0 and 1.

Write then

$$\frac{b^2}{a^2}=\sin^2\alpha,\quad \frac{a^2}{b^2}\cdot\frac{b^2-c^2}{a^2-c^2}=k^2,\quad \frac{b^2-c^2}{a^2-c^2}=k^2\sin^2\alpha, \tag{18}$$

and P takes the form

$$P=\frac{\sin\alpha\,\Delta\alpha}{\cos\alpha}\int\frac{\Delta\theta\,d\theta}{1-k^2\sin^2\alpha\sin^2\theta}, \tag{19}$$

since

$$\Delta\alpha=\sqrt{\frac{a^2-b^2}{a^2-c^2}}\text{ and }\cos\alpha=\sqrt{\frac{a^2-b^2}{a^2}}.$$

If we call ϵ the eccentricity of the section of the ellipsoid made by the plane xy, it is clear that

$$\alpha=\cos^{-1}\epsilon. \tag{20}$$

The above value for P can also be written in the form

$$P=\tan\alpha\,\Delta\alpha\int\frac{d\theta}{\Delta\theta}-k^2\sin\alpha\cos\alpha\,\Delta\alpha\int\frac{\sin^2\theta\,d\theta}{(1-k^2\sin^2\alpha\sin^2\theta)\,\Delta\theta}. \tag{21}$$

Introducing elliptic functions by means of the equations

$$t=\int\frac{d\theta}{\Delta\theta},\quad a=\int\frac{d\alpha}{\Delta\alpha}$$

(21) becomes
$$P = \text{tn}\, a \,\text{dn}\, a \,.\, t - \int \frac{k^2 \,\text{sn}\, a \,\text{cn}\, a \,\text{dn}\, a \,\text{sn}^2 t \,.\, dt}{1 - k^2 \,\text{sn}^2 a \,\text{sn}^2 t} \cdot \tag{22}$$

The quantity under the integral sign is (Cayley's Elliptic Functions, page 15) the elliptic integral of the third kind, or $\Pi\,(t\,, a)$; but introducing Jacobi's functions Z and Θ by the relation

$$\Pi\,(t\,, a) = t\,\text{Z}a + \tfrac{1}{2}\log\frac{\Theta\,(t-a)}{\Theta\,(t+a)}\,, \tag{23}$$

we are enabled to write immediately

$$P = [\text{tn}\, a \,\text{dn}\, a - \text{Z}a]\; t - \tfrac{1}{2}\log\frac{\Theta\,(t-a)}{\Theta\,(t+a)}\,, \tag{24}$$

or, writing
$$\log e^{2\,[\text{tn}\, a\, \text{dn}\, a\, -\, \text{Z}a]\, t} = \log e^{2\,wt},$$

$$P = \tfrac{1}{2}\log\frac{\Theta\,(t-a)}{\Theta\,(t+a)}\cdot e^{2\,wt}. \tag{25}$$

For the value of w observe (Cayley's Elliptic Functions, page 157) that

$$\text{H}\,(u+K) = \Theta\,(u)\,\sqrt{\frac{k}{k'}}\;\text{cn}\, u,$$

and further that
$$-\frac{d}{du}\log\,\text{cn}\, u = \frac{\text{sn}\, u\,\text{dn}\, u}{\text{cn}\, u} = \text{tn}\, u\,\text{dn}\, u$$

and
$$\frac{d}{du}\log\,\Theta\,(u) = \frac{\Theta'(a)}{\Theta\,(u)} = \text{Z}\,(u)\,.$$

We have then
$$-\frac{d}{da}\log\,\text{H}\,(a+K) = \text{tn}\, u\,\text{dn}\, u - \text{Z}\,(u)\,.$$

The functions Θ and H expressed in terms of the q-functions are

$$\Theta\left(\frac{2\,K}{\pi}\,t'\right) = 1 - 2\,q\cos 2\,\tau + 2\,q^4\cos 4\,\tau - 2\,q^9\cos 6\,\tau + \dots$$

$$\text{H}\left(\frac{2\,K}{\pi}\,t'\right) = 2\,\sqrt[4]{q}\;\{\sin\tau - q^2\sin 3\,\tau + q^6\sin 5\,\tau - q^{12}\sin 7\,\tau + \dots .$$

or, writing $\frac{2\,K}{\pi}\,t' = t$, these may be expressed as

$$\Theta\,(t) = \Theta\left(\frac{2\,K}{\pi}\,t'\right) = 1 + 2\,\Sigma\,(-1)^j\,q^{j^2}\cos 2\,jt'$$

$$\text{H}\,(t) = \text{H}\left(\frac{2\,K}{\pi}\,t'\right) = 2\,\sqrt[4]{q}\;\Sigma\,(-1)^{j-1}\,q^{j(j-1)}\sin\,(2j-1)\,t',$$

and consequently
$$\frac{\Theta\,(t+a)}{\Theta\,(t-a)} = \frac{1 + 2\,\Sigma\,(-1)^j\,q^{j^2}\cos 2\,j\,(t'+a')}{1 + 2\,\Sigma\,(-1)^j\,q^{j^2}\cos 2\,j\,(t'-a')}$$

and
$$\mathrm{H}\,(a+K)=2\sqrt[4]{q}\,\Sigma\, q^{j(j-1)}\cos\,(2j-1)\,a',$$

where
$$a'=\frac{\pi a}{2K}\,.$$

Since
$$\frac{d}{da}=\frac{\pi}{2K}\,\frac{d}{da}$$

we have for w the value
$$w=-\frac{d}{da}\log\,\mathrm{H}\,(a+K)=-\frac{\pi}{2K}\,\frac{d}{da}\log\,\mathrm{H}\,(a+K)$$
$$=\frac{\pi}{2K}\,\frac{\Sigma\,(2j-1)\,q^{j(j-1)}\sin\,(2j-1)\,a'}{\Sigma\,q^{j(j-1)}\cos\,(2j-1)\,a'}\,.$$

The complete value of P is now
$$P=\tfrac{1}{2}\log\left\{\frac{\Theta\,(t-a)}{\Theta\,(t+a)}\cdot\exp.\,\frac{\pi t}{K}\,\frac{\Sigma\,(2j-1)\,q^{j(j-1)}\sin\,(2j-1)\,a'}{\Sigma\,q^{j(j-1)}\cos\,(2j-1)\,a'}\right\},$$

exp. u standing for e^u.

If, instead of the quantities t and a, their complements are introduced as
$$K-t=t_1\,,\ K-a=a_1$$

and also
$$\operatorname{am}\,(t_1)=\theta_1\,,\ \operatorname{am}\,(a_1)=\alpha_1\,,$$

then the integral expression for P becomes
$$P=\frac{\sin\alpha\Delta\alpha}{\cos\alpha}\int\frac{\Delta\theta d\theta}{1-k^2\sin^2\alpha\,\sin^2\theta}$$
$$=\frac{\cos\alpha_1\Delta\alpha_1}{\sin\alpha_1}\int\frac{d\theta_1}{(1-k^2\sin^2\alpha_1\,\sin^2\theta_1)\,\Delta\theta_1}\,.$$

Now
$$\frac{d\theta_1}{\Delta\theta_1\,(1-k^2\sin^2\alpha_1\,\sin^2\theta_1)}=\frac{d\theta_1}{\Delta\theta_1}\left[\frac{1+k^2\sin^2\alpha_1\,\sin^2\theta_1}{1-k^2\sin^2\alpha_1\,\sin^2\theta_1}\right];$$

and therefore, by changing the order of integration and dropping a useless constant,
$$P=\frac{\cos\alpha_1\Delta\alpha_1}{\sin\alpha_1}\int\frac{d\theta_1}{\Delta\theta_1}+k^2\cos\alpha_1\,\sin\alpha_1\Delta\alpha_1\int\frac{d\theta_1}{\Delta\theta_1\,(1-k^2\sin^2\alpha_1\,\sin^2\theta_1)}$$
$$=\frac{d\log\mathrm{H}\,(a_1)}{da_1}\,t_1-\tfrac{1}{2}\log\frac{\Theta\,(t_1+a_1)}{\Theta\,(t_1-a_1)}\,.$$

If we introduce an imaginary argument and the complementary modulus, this expression takes another form rather more convenient for computation. We have (Cayley's Elliptic Functions, page 151)
$$\Theta\,(u)=\sqrt{\frac{k'K}{kK'}}\,e^{\frac{-\pi u^2}{4KK'}}\,\frac{1}{\operatorname{cn}u}\,\Theta\,(iu\,,k')\,,$$

or again
$$\Theta(u) = \sqrt{\frac{K}{K'}}\, e^{\frac{-\pi u^2}{4KK'}}\, \mathrm{H}(K' \pm iu, k')$$

and
$$\mathrm{H}(u) = -\sqrt{\frac{K}{K'}}\, e^{\frac{-\pi u^2}{4KK'}}\, \mathrm{H}(iu, k');$$

now, defining w_1 as
$$w_1 = \frac{d}{da_1} \log \mathrm{H}(ia_1, k'),$$

it is obvious that
$$w = -i\frac{\pi a_1}{2KK'} + w_1, = \frac{d}{da_1} \log \mathrm{H}(a_1).$$

Now, introducing the q-functions,
$$\mathrm{H}(ia_1, k') = 2\sqrt[4]{q'}\,\{\sin ia_1 - q_1^2 \sin 3ia_1' + q_1^6 \sin 5ia_1' \ldots\}$$

and therefore, introducing exponentials,
$$w_1 = \frac{\pi}{2K'} \cdot \frac{\Sigma(-1)^{j-1}(2j-1)\, q_1^{j(j-1)}\left[e^{(2j-1)a_1'} + e^{-(2j-1)a_1'}\right]}{\Sigma(-1)^{j-1} q_1^{j(j-1)}\left[e^{(2j-1)a_1'} - e^{-(2j-1)a_1'}\right]},$$

where $a_1' = \frac{\pi a_1}{2K'}$. The function P has already been determined as
$$P = wt_1 - \tfrac{1}{2}\log\frac{\Theta(t_1 + a_1)}{\Theta(t_1 - a_1)},$$

and can now be written as
$$P = w_1 t_1 - \tfrac{1}{2}\log\frac{\mathrm{H}(K' - i(t_1 + a_1), k')}{\mathrm{H}(K' - i(t_1 - a_1), k')}.$$

The logarithm of this ratio of conjugate H-functions is given by
$$\tfrac{1}{2}\log\frac{\Sigma q^{j(j-1)}\left[e^{(2j-1)(t_1' - a_1')} + e^{-(2j-1)(t_1' + a_1')}\right]}{\Sigma q^{j(j-1)}\left[e^{(2j-1)(t_1' - a_1')} - e^{-(2j-1)(t_1' - a_1')}\right]},$$

where $t_1' = \frac{\pi t'}{2K'}$. The final form of P is then
$$P = \frac{\pi t'}{2K'}\,\frac{\Sigma(-1)^{j-1}(2j-1)\, q^{j(j-1)}\left[e^{(2j-1)a_1'} + e^{-(2j-1)a_1'}\right]}{\Sigma(-1)^{j-1}(2j-1)\, q^{j(j-1)}\left[e^{(2j-1)a_1'} - e^{-(2j-1)a_1'}\right]}$$
$$-\tfrac{1}{2}\log\frac{\Sigma q^{j(j-1)}\left[e^{(2j-1)(t_1' - a_1')} + e^{-(2j-1)(t_1' + a_1')}\right]}{\Sigma q^{j(j-1)}\left[e^{(2j-1)(t_1' - a_1')} - e^{-(2j-1)(t_1' - a_1')}\right]}.$$

By interchanging λ_1 into λ_2 the integral expression for P becomes
$$= iQ.$$

The quantities k, a do not depend on either λ_1 or λ_2, and in consequence are unaltered by this change; the same is, of course, true of the constants a, a', w, which are functions of k, a and other constants. The only quantity which can

vary is t, and this, on account of the prescribed limits of λ_2, will become a pure imaginary, say

$$t = i(K' + \tau);$$

then

$$\theta = \operatorname{am} i(K' + \tau);$$

now

$$\operatorname{sn}^2 i(K' + \tau) = \frac{1}{k^2 \operatorname{sn}^2 i\tau}$$

and

$$\operatorname{sn}(i\tau, k) = \frac{i \operatorname{sn}(\tau, k')}{\operatorname{cn}(\tau, k')}\; i \operatorname{tn}(\tau, k');$$

but

$$\frac{1}{k^2 \operatorname{sn}^2 t} = \frac{(a^2 + \lambda_2)}{a^2} \cdot \frac{b^2}{(b^2 + \lambda_2)} = \frac{1}{k^2 \operatorname{sn}^2 i(K + \tau)};$$

therefore

$$\operatorname{tn}^2(\tau, k') = -\operatorname{sn}^2(i\tau) = \frac{b^2(a^2 + \lambda_2)}{a^2(b^2 - \lambda_2)}.$$

Writing

$$\psi = \operatorname{am}(\tau, k'),$$

we derive

$$\cos^2\psi = \frac{a^2(b^2 + \lambda_2)}{(a^2 - b^2)\lambda_2}, \quad \sin^2\psi = \frac{(a^2 + \lambda_2)b^2}{(a^2 - b^2)\lambda_2},$$

and, for the complementary modulus k',

$$k'^2 = \frac{c^2(a^2 - b^2)}{b^2(a^2 - c^2)},$$

from which follow

$$k'^2 \sin^2\psi = \frac{(a^2 + \lambda_2)c^2}{(c^2 - a^2)\lambda_2}$$

and

$$\Delta(\psi, k') = \sqrt{1 - k'^2 \sin^2\psi} = \sqrt{\frac{a^2(c^2 + \lambda_2)}{(a^2 - c^2)\lambda_2}}.$$

Now, since

$$P = wt + \log\sqrt{\frac{\Theta(t + a)}{\Theta(t - a)}}$$

and

$$Q = \frac{1}{i} P,$$

it follows at once that

$$Q = w(K' + \tau) - \frac{i}{2}\log\frac{\Theta(iK' + i\tau + a)}{\Theta(iK' + i\tau - a)};$$

but

$$\Theta(u + iK') = i e^{\frac{\pi}{4K}(K' - 2iu)}\, \mathrm{H}(u);$$

therefore

$$Q = w(K' + \tau) - \frac{i}{2}\log\frac{\mathrm{H}(a + i\tau)}{\mathrm{H}(a - i\tau)},$$

or, dropping the constant wK',

$$= w\tau - \tan^{-1}\frac{\Sigma(-1)^{j-1} q^{j(j-1)} \cos(2j-1)a' \left[e^{(2j-1)\tau'} - e^{-(2j-1)\tau'}\right]}{\Sigma(-1)^{j-1} q^{j(j-1)} \sin(2j-1)a' \left[e^{(2j-1)\tau'} + e^{-(2j-1)\tau'}\right]},$$

where
$$a' = \frac{\pi a}{2K}, \tau' = \frac{\pi \tau}{2K}.$$

On transforming Q to the complementary argument a_1 it becomes

$$Q = w\tau - \frac{1}{2i} \log \frac{\mathrm{H}\,(K - a_1 + i\tau)}{\mathrm{H}\,(K - a_1 - i\tau)}$$

$$= w_1\tau - \frac{1}{2i} \log \frac{\Theta\,(\tau + ia_1, k')}{\Theta\,(\tau - ia_1, k')},$$

or finally

$$Q = w_1\tau - \tan^{-1} \frac{\Sigma\,(-1)^{j-1} q^{j^2} \sin 2j\tau' \left[e^{2ja_1'} - e^{-2ja_1'}\right]}{1 - \Sigma\,(-1)^{j} q^{j^2} \cos 2j\tau' \left[e^{2ja_1'} - e^{-2ja_1'}\right]}.$$

We have said that it is only necessary, in order to solve the proposed problem in the most general manner, to place U equal to the real and $i \log \cot \frac{1}{2} V$ to the imaginary part of

$$f(P + iQ),$$

where f is an arbitrary functional symbol; this is the same thing as writing

$$U + i \log \cot \tfrac{1}{2} V = f(P + iQ),$$
$$U - i \log \cot \tfrac{1}{2} V = f(P - iQ).$$

Suppose that we take

$$f(\nu) = \nu,$$

then follow

$$U = P,$$
$$V = \cot^{-1} e^Q;$$

by the first of these relations all curves which depend only upon t are projected into curves depending only upon U, the longitude; that is, all the curves depending upon t are projected into meridians; but

$$t = \int \frac{d\theta}{\Delta\theta},$$

and θ is a function of λ_1 only, so that t is a function of λ_1, and the curves which depend upon t are the lines of curvature cut out of the ellipsoid by the hyperboloid of two nappes; these lines of curvature are then projected upon the sphere in the meridians, and in like manner the equation

$$V = \cot^{-1} e^Q$$

shows that the second system of lines of curvature is projected in the parallels of latitude.

If the sphere be projected into the plane $\xi\eta$ by the relations

$$(U + i \log \cot \tfrac{1}{2} V) = \log (\xi + i\eta)$$
$$(U - i \log \cot \tfrac{1}{2} V) = \log (\xi - i\eta),$$

all of the parallels will be projected in concentric circles, and all of the meridians in straight lines passing through the centre of the circles (the stereographic projection). Now, if we introduce polar co-ordinates (ρ, ϕ) by the formulæ

$$\xi = \rho \cos \phi \, , \; \eta = \rho \sin \phi ,$$

we have

$$\xi^2 + \eta^2 = \rho^2 ;$$

but

$$\log (\xi + i\eta) + \log (\xi - i\eta) = \log (\xi^2 + \eta^2) = \log \rho^2$$

and

$$\log (\xi + i\eta) - \log (\xi - i\eta) = \log \frac{\cos \phi + i \sin \phi}{\cos \phi - i \sin \phi} = 2\, i\phi ;$$

therefore

$$U = \log \rho ,$$
$$V = \cot^{-1} e^{\phi} ;$$

but

$$U = P ,$$
$$V = \cot^{-1} e^{Q} ,$$

consequently

$$\rho = e^P = e^{wt} \sqrt{\frac{\Theta\,(t + a)}{\Theta\,(t - a)}} ,$$

$$\phi = Q = wt + \frac{1}{2i} \log \frac{\mathrm{H}\,(a + i\tau)}{\mathrm{H}\,(a - i\tau)} .$$

We have in this case the lines of curvature on the ellipsoid arising from its intersections with the hyperboloid of two nappes projected on the plane in straight lines passing through the origin of co-ordinates; the remaining system of lines of curvature being projected in concentric circles whose common centre is at the origin of co-ordinates. If the ellipsoid be one of revolution around the axis of x, then result

$$b = c \, , \; k = o \, , \; k' = 1 \, , q = o \, , \; K = \frac{\pi}{2} \, , \; K' = o ,$$
$$\theta = t = t' \, , \; a = \alpha = \alpha' \, , \; w = \tan \alpha ,$$

$$\tau' = \tau = \int_0^{\psi} \frac{d\psi}{\cos \psi} = \log \tan \tfrac{1}{2} (90^\circ + \psi)$$

and

$$\frac{e^{\tau'} - e^{-\tau'}}{e^{\tau'} + e^{-\tau'}} , = \sin \psi .$$

For the position of the point x, y, z, when projected on the sphere, we have

$$U = \tan \alpha \, . \, \theta ,$$
$$V = \tan \alpha \, . \log \tan \tfrac{1}{2} \, (90^\circ + \psi) + \tan^{-1} \left[\cot \alpha \sin \phi\right] ,$$

[illegible]

[illegible]

$$[illegible]$$

[illegible] terms that we obtained for [illegible]

[illegible]

$$[illegible]$$

[illegible] determined [illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible] remains only to give the values [illegible] in terms of the same [illegible] We had

$$[illegible]$$

now

$$a^2 + \lambda_1 = \frac{a^2 - b^2}{D_1}, \qquad a^2 + \lambda_2 = \frac{(a^2 - b^2)\sin^2\psi}{D_2},$$

$$b^2 + \lambda_1 = \frac{(a^2 - b^2)\sin^2\alpha\sin^2\theta}{D_1}, \qquad b^2 + \lambda_2 = \frac{(a^2 - b^2)\sin^2\alpha\cos^2\psi}{D_2},$$

$$c^2 + \lambda_1 = \frac{(a^2 - c^2)\sin^2\alpha\cos^2\theta}{D_1}, \qquad c^2 + \lambda_2 = \frac{(a^2 - c^2)\sin^2\alpha\,\Delta^2(\psi, k')}{D_2},$$

where

$$D_1 = 1 - k^2\sin^2\alpha\sin^2\theta,$$
$$D_2 = \sin^2\alpha + \cos^2\alpha\sin^2\psi,$$

or, introducing the notation of elliptic functions,

$$\theta = \operatorname{am} t, \ \psi = \operatorname{am}(\tau, k'),$$
$$D_1 = 1 - \operatorname{sn}^2 a \operatorname{sn}^2 t,$$
$$D_2 = 1 - \operatorname{sn}^2 a \operatorname{sn}^2(\tau, k'),$$

and

$$a^2 + \lambda_1 = \frac{a^2 - b^2}{D_1}, \qquad a^2 + \lambda_2 = \frac{(a^2 - b^2)\operatorname{sn}^2(\tau, k')}{D_2},$$

$$b^2 + \lambda_1 = \frac{(a^2 - b^2)}{D_1}\operatorname{sn}^2 a \operatorname{sn}^2 t, \qquad b^2 + \lambda_2 = \frac{(a^2 - b^2)\operatorname{sn}^2 a \operatorname{cn}^2(\tau, k')}{D_2},$$

$$c^2 + \lambda_1 = \frac{(a^2 - c^2)}{D_1}\operatorname{sn}^2 a \operatorname{cn}^2 t, \qquad c^2 + \lambda_2 = \frac{(a^2 - c^2)\operatorname{sn}^2 a \operatorname{dn}^2(\tau, k')}{D_2}.$$

Observing now the relations

$$\frac{a^2 - b^2}{a^2 - c^2} = \Delta^2\alpha, \quad \frac{(a^2 - b^2)\,b^2}{b^2 - c^2} = \frac{a^2\,\Delta^2\alpha}{k^2}, \quad \frac{(a^2 - c^2)\,c^2}{b^2 - c^2} = \frac{a^2\,k'^2}{\Delta^2\alpha\,.\,k^2},$$

and substituting in these $\operatorname{dn} a$ for $\Delta\alpha$, we are enabled to write x, y, z, in the forms

$$x = G_1\,.\operatorname{dn}^2 a \operatorname{sn}(\tau, k'),$$
$$y = G_1\,.\operatorname{dn}^2 a \operatorname{sn}^2 a \operatorname{sn}\theta \operatorname{cn}(\tau, k'),$$
$$z = G_1\,.\,k' \operatorname{sn}^2 a \operatorname{cn}\theta \operatorname{dn}(\tau, k'),$$

where

$$G_1 = \frac{a}{\operatorname{dn} a}\,\frac{1}{\sqrt{(1 - k^2\operatorname{sn} a \operatorname{sn}\theta)(\operatorname{sn}^2 a + \operatorname{cn}^2 a \operatorname{sn}^2)(\tau, k')}};$$

the a in the numerator of course denotes the semi-axis major of the ellipsoid. The angle α has been defined by

$$\sin^2\alpha = \frac{b^2}{a^2};$$

from this

$$b^2 = a^2\sin^2\alpha,$$

and we also obtain quite readily

$$c^2 = \frac{a^2\,k'^2\sin^2\alpha}{\Delta^2\alpha};$$

where
$$G_3 = \frac{a}{\sqrt{1 - \operatorname{sn}^2 a_1 \operatorname{sn}^2 t_1}}.$$

The angle ψ is here the longitude and θ_1 the excentric anomaly of the meridians. The rectangular co-ordinates ξ, η, ζ, can of course be given in terms of (t, a, τ) or (t_1, a_1, τ) by means of the relations

$$\xi = R \cos U \sin V,$$
$$\eta = R \sin U \sin V,$$
$$\zeta = R \cos V;$$

in terms of P and Q these become

$$\xi = \frac{R \cos P}{\sqrt{e^{2Q} + 1}},$$
$$\eta = \frac{R \sin P}{\sqrt{e^{2Q} + 1}},$$
$$\zeta = \frac{Re^{2Q}}{\sqrt{e^{2Q} + 1}}.$$

On the Calculation of the Generating Functions and Tables of Groundforms for Binary Quantics.

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The object of this paper is to give an account of the methods, due to Professors Cayley and Sylvester, of calculating the generating functions pertaining to binary quantics and thence determining the number of fundamental invariants and covariants of any order and degree. As it will not very greatly increase the length of the paper, I shall endeavor, besides giving the processes of calculation, to present a connected view, though not a complete discussion, of the subject. It seems desirable to make some remarks at the outset on the terms employed, though these are, for the most part, well understood.

The *degree* of any function is its degree in the coefficients of the quantic, the *order* is its degree in the variables. The term *covariant* will be regarded, whenever convenient, as including invariants, the latter being covariants of the order zero. A *differentiant* is a symmetric function of the differences of the roots; since the source of any covariant (the coefficient of the highest power of x in the covariant) is a differentiant, and since the covariant is completely determined by its source, the discovery of covariants is reduced to that of differentiants. A *groundform* is an irreducible or fundamental invariant or covariant, i. e. one that is not a rational integral function of invariants and covariants of lower degrees and orders.

The symbol $(w:i,j)$ is employed to denote the number of ways in which the number w can be composed as the sum of j of the numbers $0,1,2,\ldots\ldots i$ (repetitions being allowed); or, in other words, the number of ways in which w can be composed of j or fewer positive integers none greater than i. Putting $g = ij - 2w$, it will often be convenient to write $(i,j:g)$ instead of the above symbol, so that we have

$$(w:i,j) = (i,j:ij-2w); \qquad (i,j:g) = \left(\frac{ij-g}{2}:i,j\right).$$

SINGLE QUANTICS.

For simplicity, we shall first consider exclusively the case of a single quantic. The methods are all based upon the following fundamental theorem:

The number of linearly independent differentiants of the weight w and degree j, belonging to a quantic of the order i, is $(w:i,j)-(w-1:i,j)$. We shall use $\Delta(w:i,j)$ to denote $(w:i,j)-(w-1:i,j)$.

A differentiant of weight w and degree j, belonging to a quantic of order i, is the source of a covariant whose order is $ij-2w$, say g; so that the above theorem may be thus stated:

The number of linearly independent covariants of the order g and degree j, belonging to a quantic of the order i, is $(i,j:g)-(i,j:g+2)$. We shall use $\Delta(i,j:g)$ to denote $(i,j:g)-(i,j:g+2)$.

$(w:i,j)$ is the coefficient of $c^j z^w$ in the development of

$$\frac{1}{(1-c)(1-cz)(1-cz^2)\dots(1-cz^i)} \tag{1}$$

in ascending powers of c; or, putting $z=x^{-2}$, $c=ax^i$, we may say that $(w:i,j)$, $=(i,j:g)$, is the coefficient of a^jx^g in the development of

$$\frac{1}{(1-ax^i)(1-ax^{i-2})\dots(1-ax^{-i+2})(1-ax^{-i})} \tag{2}$$

in ascending powers of a;
hence $\Delta(i,j:g)$ is the coefficient of a^jx^g in the development of

$$\frac{1-x^{-2}}{(1-ax^i)(1-ax^{i-2})\dots(1-ax^{-i+2})(1-ax^{-i})} \tag{3}$$

in ascending powers of a.

From (1) is also deduced Euler's theorem that $(w:i,j)$ is the coefficient of z^w in the development of

$$\frac{(1-z^{j+1})(1-z^{j+2})\dots(1-z^{j+i})}{(1-z)(1-z^2)\dots(1-z^i)}, \tag{4}$$

so that $\Delta(w:i,j)$ is the coefficient of z^w in the development of

$$\frac{(1-z^{j+1})(1-z^{j+2})\dots(1-z^{j+i})}{(1-z^2)\dots(1-z^i)}. \tag{5}$$

The fraction (3) is a generating function in which the coefficient of a^jx^g is the number of linearly independent covariants of the degree j and order g, to a quantic of the order i; but it is far from being well adapted to calculation, and moreover furnishes no means of determining *the complete system of groundforms* of the quantic. Before describing the methods of constructing generating functions

which *do* serve this important purpose, it will be well to make some remarks upon the connection between the determination of the groundforms and that of the number of linearly independent covariants of a given degree and order.

Tamisage.

To begin with an example: Suppose we have found that the quintic has one irreducible invariant of each of the degrees 4, 8, 12, and no other irreducible invariant of a degree lower than 16; and let it be proposed to find the number of irreducible invariants of the degrees 16 and 18. We can find by actually considering the partitions, or otherwise, that $\Delta(5, 16:0) = 4$; hence there are four linearly independent invariants of degree 16. Now, precisely four invariants of degree 16 can be formed from the lower irreducible invariants (which latter we may denote as (4.0), (8.0) and (12.0)), namely, $(4.0)^4$, $(4.0)^2(8.0)$, (4.0) (12.0) and $(8.0)^2$; hence, unless we suppose the previously found irreducible invariants to be connected by a syzygetic relation of degree 16, we conclude that there is no new — i. e. no irreducible — invariant of degree 16. Also $\Delta(5, 18:0) = 1$, so that there is one invariant of degree 18; and since no invariant of that degree can be formed from the lower invariants, we *know* that there is an irreducible invariant of degree 18. We should have made the same inference if $\Delta(5, 18:0)$ had not been 1, but had exceeded by 1 the number of ways in which invariants of degree 18 could be formed from lower irreducible invariants, provided we assumed, as before, that those lower invariants were not connected by any syzygetic relation of the degree in question. Again, keeping to the example of the quintic, suppose we have found that there is one irreducible covariant of each of the following types:

(4.0), (8.0), (5.1), (7.1), (2.2), (6.2), (8.2), (3.3), (5.3), (9.3), (4.4), (6.4)

where $(m.n)$ represents a covariant of degorder $(m.n)$, and that there are no other irreducible covariants whose degree is less than 10 and order not greater than 4; and let it be proposed to find the number of irreducible covariants of degree 10, order 4. We find $\Delta(5, 10:4) = 3$, so that there are three linearly independent covariants of degorder (10.4); and covariants of this degorder can be formed by compounding lower irreducibles in the following four ways:

(4.0) (6.4), (5.1) (5.3), (7.1) (3.3), (2.2) (8.2).

In this case, then, the number of compound forms exceeds by 1 the total number of linearly independent forms of the type in question, (10.4); hence there must be at least *one* syzygy of degorder (10.4) connecting the lower groundforms:

but we are not compelled to assume that *more* than one such syzygy exists; and assuming that the one *necessary* syzygy is the *only* one, we conclude that the three linearly independent covariants of degorder (10.4) are compounded ones, and that there is no groundform of that order.

Stated in general terms, the matter stands as follows: By means of the fundamental theorem,* we can ascertain the number (say α) of *linearly independent* covariants of degree j and order g; suppose now that we also know the number of *irreducible* covariants of every type whose degree is below j and whose order is not higher than g; and suppose there are β ways of producing a covariant of the given type (degorder $(j.g)$) by multiplying together these irreducible covariants. Then, if β is less than α, there are evidently *at least* $\alpha-\beta$ irreducible covariants of the given type; if β is equal to or greater than α, there *may* not be any irreducible covariants of the given type. In fact, if the β possible compound forms of the type are linearly independent, there remain $\alpha-\beta$ forms which are *not* compounds, i. e. which are groundforms; if β is greater than α, the β forms can not *all* be linearly independent; but α of them may be independent, and if they are, no uncompounded forms of the type remain. We assume (in the absence of demonstration, but with the support of very strong inductive evidence) that there never are groundforms and syzygies of one and the same degorder. Granting this fundamental postulate, the number of groundforms of the type considered is $\alpha-\beta$ if $\alpha>\beta$; if $\alpha \overline{\lessdot} \beta$ the number is zero. Thus, if the numbers of linearly independent covariants of every degree up to j and of every order up to g be known, the numbers of the groundforms within the same limits are obtained by a process which Professor Sylvester calls *tamisage*, and of which it may be desirable to give an example.

Take the case of the octavic; the numbers of linearly independent forms of all degrees to the 7th and of all orders to the 8th are shown in the first of the following tables; the numbers of groundforms within the same limits are shown in the second table. The second table is deduced from the first as follows: We proceed regularly down the successive columns, writing in the corresponding places of the second table the number of groundforms of each type. Thus, (2.0) and (3.0) are groundforms, there being no lower forms out of which they can be compounded; one (4.0) can be produced by multiplying (2.0) by itself, so that the number of groundforms of this type is $2-1=1$; (2.0) (3.0) giving (5.0), we have again $2-1=1$; $(2.0)^3$, (2.0) (4.0), and $(3.0)^2$ each give one (6.0), so that the number of groundforms of this type is $4-3=1$; $(2.0)^2$ (3.0), (2.0) (5.0), (3.0) (4.0)

* Which, it is to be remembered, has been rigorously demonstrated by Professor Sylvester. See Borchardt's Journal, Vol. LXXXV. (1878), p. 104, and Philos. Mag., Vol. V. (1878), p. 178.

each giving (7.0), we have $4 - 3 = 1$ groundform of this type. We then proceed to the next column, deducting, as before, from each number in the first table the number of compound forms of the corresponding type that can be produced by the previously found groundforms; so that at any stage of the process we know

LINEARLY INDEPENDENT FORMS.

Degree \ Order	0	2	4	6	8
1					1
2	1		1		1
3	1		1	1	2
4	2		3	1	4
5	2	1	4	3	6
6	4	1	7	5	11
7	4	3	10	9	16

GROUNDFORMS.

Degree \ Order	0	2	4	6	8
1					1
2	1		1		1
3	1		1	1	1
4	1		2	1	1
5	1	1	2	2	1
6	1	1	2	3	1
7	1	2	2	3	0

the number of linearly independent forms of a certain type and the number of groundforms of all lower types. Thus, suppose we have determined all the numbers in the second table except the last one, this is found as follows:

$$(2.0)\,,(2.0)^2,(2.0)^3,(2.0)\,(3.0)\,,(2.0)\,(4.0)\,,(3.0)\,,(3.0)^2,\,(4.0)\,,(5.0)\,,(6.0)$$

combined respectively with

$$(5.8)\,,(3.8)\;\;,(1.8)\;,\quad(2.8)\quad,\quad(1.8)\quad,\;(4.8),(1.8)\;,(3.8)\,,(2.8)\,,(1.8)$$

give 10 forms of the type (7.8), there being one groundform of each of the types above employed; $(3.0)t(2.4)^2$ and (2.0) (2.4) (3.4) likewise give 2; while the groundform (2.4) combined with each of the two groundforms (5.4), and the groundform (3.4) combined with each of the two groundforms (4.4) give in all 4 forms of the type (7.8). Thus, the total number of compound forms is $10 + 2 + 4 = 16$, and 16 being also the number of linearly independent forms, we see that there are no groundforms of the type in question.

This process of tamisage is evidently very long and tedious, the above example being of the simplest, and the labor plainly becoming greater at every step; moreover, however far we may carry the process we have no assurance that there are no groundforms beyond: the series representing the numbers of linearly independent forms being evidently infinite in extent. With the generating functions we are about to consider, the labor of tamisage is very greatly abridged, and, what is of essential importance, the field in which it has to be applied is finite.

Sylvester's First Method.

The above "*crude form*" (3) of the generating function is

$$\phi(x) = \frac{1 - x^{-2}}{(1 - ax^i)(1 - ax^{i-2}) \ldots . (1 - ax^{-i+2})(1 - ax^{-i})};$$

consider the decomposition of this into partial fractions, with reference to x. To any factor $1 - ax^\lambda$ of the denominator, λ being positive, will correspond λ partial fractions of the form

$$\frac{A}{1 - \rho a^{\frac{1}{\lambda}} x},$$

where A is a function of a, and ρ a λth root of unity; these λ fractions added together will give a fraction of the form

$$\frac{A_0 + A_1 x + A_2 x^2 + \ldots . + A_{\lambda-1} x^{\lambda-1}}{1 - ax^\lambda},$$

expanding which in ascending powers of a, we get also ascending powers of x, beginning with the 0th power. But to any factor $1 - ax^{-\lambda}$ will correspond λ partial fractions of the form

$$\frac{A}{x - \rho a^{\frac{1}{\lambda}}};$$

the sum of these λ fractions is of the form

$$\frac{A_0 + A_1 x + A_2 x^2 + \ldots . + A_{\lambda-1} x^{\lambda-1}}{x^\lambda - a},$$

i. e.
$$\frac{1}{x^\lambda} \cdot \frac{A_0 + A_1 x + A_2 x^2 + \ldots . + A_{\lambda-1} x^{\lambda-1}}{1 - ax^{-\lambda}},$$

and this, when expanded in ascending powers of a, gives only negative powers of x, which are irrelevant to the question in hand. We can therefore obtain a generating function whose development shall coincide with that of $\phi(x)$ as far as the terms containing non-negative powers of x are concerned, by calculating the partial fractions corresponding to those factors of the denominator of $\phi(x)$ in which the exponent of x is positive, and taking the sum of these partial fractions.

The fraction corresponding to $1 - ax^\lambda$ is, as before stated,

$$\Sigma \frac{A}{1 - \rho a^{\frac{1}{\lambda}} x},$$

where ρ is to be successively every λth root of unity. Denoting by $\phi_\lambda(x)$ what $\phi(x)$ becomes when the factor $1 - ax^\lambda$ is struck out of the denominator, and writing α for $\rho a^{\frac{1}{\lambda}}$, it is seen at once that the value of A is $\frac{1}{\lambda}\phi_\lambda(\alpha^{-1})$; i. e.

$$\lambda A = \frac{1 - a^2}{(1 - a^{-i+\lambda})(1 - a^{-i+2+\lambda}) \dots (1 - a^{-2})(1 - a^2) \dots (1 - a^{i+\lambda})}$$

$$= (-)^{\frac{i-\lambda}{2}} \frac{a^{\frac{i-\lambda}{2} \cdot \frac{i-\lambda+2}{2}} (1 - a^2)}{(1 - a^{i-\lambda})(1 - a^{i-2-\lambda}) \dots (1 - a^2)(1 - a^2) \dots (1 - a^{i+\lambda})}$$

$$= (-)^{\frac{i-\lambda}{2}} \frac{a^{\frac{i-\lambda}{2} \cdot \frac{i-\lambda+2}{2}} (1 - a^2)}{(1-a^2)^2(1-a^4)^2(1-a^6)^2 \dots (1-a^{i-\lambda})^2(1-a^{i-\lambda+2})(1-a^{i-\lambda+4}) \dots (1-a^{i+\lambda})}.$$

Any factor $1 - a^m$ of this denominator is a divisor of $1 - a^{\lambda k}$, where λk is the least common multiple of m and λ, giving a quotient $1 + a^m + a^{2m} + \dots$; multiplying numerator and denominator by the like quotients for all the factors, the denominator becomes a product of factors of the form $1 - a^k$, and the numerator alone involves a. Thus we have

$$\frac{A}{1 - ax} = \frac{1}{\lambda} \cdot \frac{c_0 + c_1 a + c_2 a^2 + \dots}{(1 - ax)(1 - a^k)(1 - a^{k'}) \dots}$$

and hence

$$\Sigma \frac{A}{1 - ax} = \frac{1}{\lambda} \cdot \frac{\Sigma (c_0 + c_1 a + c_2 a^2 + \dots)(1 + ax + a^2x^2 + \dots + a^{\lambda-1}x^{\lambda-1})}{(1 - ax^\lambda)(1 - a^k)(1 - a^{k'}) \dots}$$

It only remains to collect those terms of the numerator in which the exponent of a is a multiple of λ (the others disappearing in the summation), and we have $\Sigma \frac{A}{1 - ax}$ expressed in the form

$$\frac{L_0 + L_1 x + \dots + L_{\lambda-1} x^{\lambda-1}}{(1 - ax^\lambda)(1 - a^k)(1 - a^{k'}) \dots}.$$

The sum of the several fractions obtained in this way is evidently of the form

$$\frac{C_0 + C_1 x + C_2 x^2 + \dots}{(1 - ax^i)(1 - ax^{i-2}) \dots (1 - a^k)(1 - a^{k'}) \dots},$$

where, it may be remarked in passing, the numerator is of lower degree in x than the denominator. This fraction, written in its lowest terms,* is called the *reduced form* of the generating function.

Representative Form of the Generating Function.

In general the reduced form does not serve directly for obtaining the numbers of the groundforms. It would serve this purpose — as we shall presently show — if the factors of the denominator all corresponded to groundforms, i. e. if to each factor $1 - a^r x^s$ (where s may be 0) in the denominator there corre-

* Not always *strictly* in its lowest terms, but in the lowest form having only factors of the types $1 - a^r$, $1 - a^r x^s$ in the denominator.

sponded a groundform of degorder $(r.s)$; and in this case the factors of the denominator are said to be representative. If this is not the case, we multiply numerator and denominator by such factors as will make the denominator a product of representative factors exclusively; such factors exist for all the quantics whose generating functions have been calculated, with the exception of the septimic, in which there is one factor in the denominator which cannot be converted into a representative one. Nor is it difficult to find these factors. Those containing x are disposed of at once; for $1 - ax^i$ represents the quantic itself, and there being no other covariant of the first degree, the factors $1 - ax^{i-2}$, $1 - ax^{i-4}$, etc. are not representative; but since every quantic has a series of covariants (obviously irreducible) of the second degree and of orders $2(i-2)$, $2(i-4)$, etc. the factors $1 - ax^{i-2}$, $1 - ax^{i-4}$, etc. will be converted into representative ones by multiplication by $1 + ax^{i-2}$, $1 + ax^{i-4}$, etc. As to any factor $1 - a^r$ independent of x, it is easy to infer from the generating function itself whether it is representative or not, i. e. whether there is or is not an irreducible *in*variant of the degree r;* and if not, we have to find whether there is one of a degree that is a multiple of r, say mr; in which case we multiply numerator and denominator by $\frac{1-a^{mr}}{1-a^r}$. For all the quantics that have been considered (with the single exception above mentioned), it is found that the non-representative factors become representative on merely *doubling* the exponent.—When these multiplications have been performed, so that every factor in the denominator is representative, the generating function is said to be in the *representative form*.†

Mode of obtaining the Table of Groundforms.

We shall now show that when the generating function is in a representative form, the groundforms consist of those represented by the factors of the denominator, together with those obtained by tamisage (with a certain modification) upon the numerator.

Let $L = \Sigma m_{j,g} a^j x^g$, where $m_{j,g}$ is the number of linearly independent covariants of degorder $(j.g)$. Then if there be a covariant, say V, of degorder $(r.s)$, the number of linearly independent covariants of degorder $(j.g)$ in which

* If a factor is repeated, it must represent a distinct groundform each time; e. g. if the denominator contains $(1 - a^6)^2$ it is not representative unless there are at least *two* irreducible invariants of the degree 6.

† Or, rather, *a* representative form. Others may be obtained by multiplying numerator and denominator by representative factors.

V enters as a factor is evidently exactly equal to the whole number of linearly independent covariants of degorder $(j-r \,.\, g-s)$, i. e. equal to the coefficient of $a^j x^g$ in $a^r x^s L$. Hence the excess of the whole number of linearly independent covariants of degorder $(j \,.\, g)$ over the number of such covariants in which V enters as a factor is the coefficient of $a^j x^g$ in $(1-a^r x^s)\, L$. It follows hence that if V is a groundform, the *other* groundforms can be got by tamisage from $(1-a^r x^s)\, L$, just as *all* the groundforms would be got from L.

In extending this result, so that we may be enabled to multiply L by more than one factor of the form $1-a^r x^s$, it is necessary to appeal to the fundamental postulate, that if there are syzygies of any given degorder, there are no groundforms of that degorder (p. 131, l. 17). Consider the linearly independent covariants of degorder $(j \,.\, g)$; suppose that the groundforms of the quantic are not connected by any syzygies of this degorder; and imagine these covariants distributed, with reference to any k groundforms $V_1, V_2, V_3 \ldots\ldots V_k$, of degorders $(r_1 \,.\, s_1), (r_2 \,.\, s_2), (r_3 \,.\, s_3) \ldots\ldots (r_k \,.\, s_k)$ into classes, as follows:—

m_k covariants containing the product of all the k V's as a factor.*
m_{k-1} " " any $(k-1)$ and only $(k-1)$ V's.
. . . .
. . . .
. . . .
m_2 covariants containing two and only two V's.
m_1 " " one " " one V.
m_0 " " no V.

Then it is easily seen that the coefficient of $a^j x^g$

$$\begin{array}{llllllll}
\text{in } L & \text{is } m_0 + & m_1 + & m_2 + & m_3 + \ldots + & m_k \\
\text{in } -\Sigma a^r x^s L & \text{is} & -\frac{1}{1} m_1 - & \frac{2}{1} m_2 - & \frac{3}{1} m_3 - \ldots - & \frac{k}{1} m_k \\
\text{in } \ \Sigma a^{r+r'} x^{s+s'} L & \text{is} & & \frac{2.1}{1.2} m_2 + & \frac{3.2}{1.2} m_3 + \ldots + & \frac{k(k-1)}{1.2} m_k \\
\text{in } -\Sigma a^{r+r'+r''} x^{s+s'+s''} L & \text{is} & & & -\frac{3.2.1}{1.2.3} m_3 - \ldots - & \frac{k(k-1)(k-2)}{1.2.3} m_k \\
 & \text{etc.} & & & \text{etc.} &
\end{array}$$

For the coefficient in L is the total number of linearly independent covariants of degorder $(j \,.\, g)$; the coefficient in $\Sigma a^r x^s L$ is the number of linearly independent covariants which multiplied by any V give rise to a covariant of

* Whether any V appears to the first or to a higher power is indifferent to this argument throughout.

degorder $(j.g)$, so that in this coefficient every such compounded covariant is counted once for each distinct V that it contains; the coefficient in $\Sigma a^{r+r'}x^{s+s'}L$ is the number of linearly independent covariants which multiplied by any two distinct V's give rise to a covariant of degorder $(j.g)$, so that in this coefficient every such compounded covariant is counted once for each distinct binary combination of the V's that it contains; and so on.

By addition of these results, we see that m_0 is the coefficient of $a^j x^g$ in

$$L\,(1 - \Sigma a^r x^s + \Sigma a^{r+r'}x^{s+s'} - \Sigma a^{r+r'+r''}x^{s+s'+s''} + \ldots\ldots)$$
$$= L\,(1 - a^{r_1}x^{s_1})\,(1 - a^{r_2}x^{s_2})\,(1 - a^{r_3}x^{s_3}) \ldots\ldots (1 - a^{r_k}x^{s_k});$$

i. e. *if the groundforms of the quantic are not connected by any syzygies of the degorder* $(j.g)$, *the coefficient of* $a^j x^g$ *in* $L(1 - a^{r_1}x^{s_1})\,(1 - a^{r_2}x^{s_2}) \ldots\ldots (1 - a^{r_k}x^{s_k})$ *is the number of linearly independent covariants of degorder* $(j.g)$ *containing none of the groundforms* $V_1, V_2, \ldots\ldots V_k$.

If there *are* syzygies of the degorder $(j.g)$, we have to distinguish between two cases. (For brevity, we shall denote $L(1 - a^{r_1}x^{s_1})\,(1 - a^{r_2}x^{s_2}) \ldots\ldots (1-a^{r_k}x^{s_k})$ by Λ.)

1° Let us suppose that no syzygies existed among covariants of lower degorders which are raised to the degorder $(j.g)$ by multiplication by groundforms. Then the coefficient of $a^j x^g$ in L has been diminished by what would be the number of covariants obtained by compounding with the V's if these compounds were linearly independent; so that if they are *not* all linearly independent the diminution has been too great: in other words, the coefficient of $a^j x^g$ is either equal to or less than the number of linearly independent covariants of degorder $(j.g)$ containing none of the V's. Now if we were to perform the operation of tamisage upon L, the existence of syzygies of the degorder $(j.g)$ would (according to the fundamental postulate) be indicated by the coefficient of $a^j x^g$ being rendered negative by the operation. Likewise, if the coefficient of $a^j x^g$ were exactly equal to the number of linearly independent covariants of degorder $(j.g)$ not containing the V's, it would be rendered negative by tamisage upon the coefficients in Λ; and *à fortiori* the *actual* coefficient (which has just been proved to be less than if not equal to the above number) will be rendered negative. Hence in this first case tamisage upon Λ gives no groundforms of degorder $(j.g)$, which is correct; and, moreover, indicates the existence of syzygies of that degorder.

2° Suppose that syzygies of the degorder $(j.g)$ exist which are not groundsyzygies, but are obtained from lower syzygies through multiplication by groundforms or by combinations of groundforms. Then, from what has been

said under 1°, the process of tamisage must have revealed these lower syzygies at their inception. We have, therefore, only to see whether $(j \,.\, g)$ is a degorder which can be formed by composition of a degorder in which a syzygy has been found to occur, with the degorder of any groundform or combination of groundforms (*not excepting the* V*'s*); if it is, we infer, by the fundamental postulate, that no groundform of the degorder $(j \,.\, g)$ exists.

Let us now return to the representative generating function. This is of the form

$$\frac{N}{(1 - a^{r_1}x^{s_1})\,(1 - a^{r_2}x^{s_2}) \ldots . (1 - a^{r_k}x^{s_k})}$$

(some of the s's are 0) where the factors of the denominator correspond to groundforms. The development of this fraction is identical with what we have above called L, so that N is identical with $L\,(1 - a^{r_1}x^{s_1})\,(1 - a^{r_2}x^{s_2}) \ldots . (1 - a^{r_k}x^{s_k})$. The foregoing discussion shows that the groundforms additional to those represented in the denominator are to be found as follows:—

Operate by tamisage upon the coefficients of N.

a) If the coefficient of a^jx^g, as reduced by tamisage, is a positive number or 0 (and provided the coefficient is not rejected in virtue of b)), this reduced coefficient is the number of groundforms of degorder $(j \,.\, g)$* and there are no syzygies of that degorder.

b) If the coefficient of a^jx^g, as reduced by tamisage, is negative, there are syzygies and no groundforms of the degorder $(j \,.\, g)$; moreover, there are syzygies and no groundforms in all higher degorders derivable from $(j \,.\, g)$ by compounding $(j \,.\, g)$ with the degorders of groundforms (the groundforms represented in the denominator, as well as those found from the numerator), so that the coefficients corresponding to such higher degorders are to be rejected.

The numerator being finite, the above operation of modified tamisage is finite, which is a matter of essential importance. The practical advantage in having to operate upon the numbers appearing in N, which are of necessity very much smaller than those in the infinite series, is obvious.

Constitution of the Generating Function as first obtained.

We shall now go back and look more closely at the form of the generating function as primarily given by Sylvester's first method; for an examination of

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* Additional, of course, to any of the same degorder that may be represented in the denominator.

this form gives rise to two other methods differing materially from the former and from each other.

It will be best to mention, in the first place, a distinction between quantics of even and quantics of odd orders which has hitherto been passed over for the sake of uniformity. For even quantics the crude form, $\phi(x)$, involves only even powers of x, and may be treated as a function of x^2, which of course greatly abridges the work. In examining the form of the generating function, we shall treat separately quantics of even and quantics of odd orders.

Let us take, first, a quantic of even order, $2n$, so that

$$\phi(x) = \frac{1 - x^{-2}}{(1 - ax^{2n})(1 - ax^{2n-2}) \ldots (1 - a) \ldots (1 - ax^{-2n+2})(1 - ax^{-2n})}$$

$$= \frac{1 - u^{-1}}{(1 - au^{n})(1 - au^{n-1}) \ldots (1 - a) \ldots (1 - au^{-n+1})(1 - au^{-n})}.$$

As before, we have to find the sum of the partial fractions corresponding to those factors of the denominator in which the exponent of u is positive. The fraction corresponding to $1 - au^{\lambda}$ is

$$\Sigma \frac{A}{1 - \rho a^{\frac{1}{\lambda}} u},$$

and we find, in the same manner as before, denoting $\rho a^{\frac{1}{\lambda}}$ by α, that

$$\lambda A = (-)^{n-\lambda} \frac{\alpha^{\frac{(n-\lambda)(n-\lambda+1)}{2}}(1 - \alpha)}{(1 - \alpha)^2(1 - \alpha^2)^2(1 - \alpha^3)^2 \ldots (1 - \alpha^{n-\lambda})^2(1 - \alpha^{n-\lambda+1}) \ldots (1 - \alpha^{n+\lambda})}.$$

Here λ is some number between 1 and n (both inclusive). When $\lambda = n$, the above denominator (after cancelling the factor $1 - \alpha$) is

$$(1 - \alpha^2)(1 - \alpha^3) \ldots (1 - \alpha^{2n-1})(1 - \alpha^2),$$

which is evidently contained in

$$(1 - \alpha^2)^2(1 - \alpha^3) \ldots (1 - \alpha^{2n-1});$$

when λ is less than n, the denominator (after cancellation of $1 - \alpha$) is

$$(1 - \alpha^2)(1 - \alpha^3) \ldots (1 - \alpha^{n+\lambda})(1 - \alpha)(1 - \alpha^2) \ldots (1 - \alpha^{n-\lambda}),$$

and this is also contained in

$$(1 - \alpha^2)^2(1 - \alpha^3) \ldots (1 - \alpha^{2n-1}).$$

For $(1 - \alpha^2)(1 - \alpha^3) \ldots (1 - \alpha^{n+\lambda})$

is contained in $\qquad (1 - \alpha^2)(1 - \alpha^3) \ldots (1 - \alpha^{n+\lambda});$

and $(1 - \alpha)(1 - \alpha^2) \ldots (1 - \alpha^{n-\lambda})$,

being contained in $(1 - a^{n+\lambda})(1 - a^{n+\lambda+1}) \ldots . (1 - a^{2n-1})$

and also in $(1 - a^{n-\lambda})(1 - a^{n+\lambda+1}) \ldots . (1 - a^{2n-1})$,

is contained in their difference, and therefore in

$$(1 - a^{2\lambda})(1 - a^{n+\lambda+1}) \ldots . (1 - a^{2n-1}),$$

which is itself contained in $(1 - a^2)(1 - a^{n+\lambda+1}) \ldots . (1 - a^{2n-1})$.

Thus we see that the partial fractions will have for a common denominator

$$(1 - a^2)^2(1 - a^3)(1 - a^4) \ldots . (1 - a^{2n-1})(1 - ax^2)(1 - ax^4) \ldots . (1 - ax^{2n}).$$

It remains to notice the degree of the numerator in a and x. The degree in x is necessarily less by at least 2 (since only even powers appear) than the degree of the denominator, and therefore cannot exceed $n(n+1) - 2$. We shall show presently (see p. 141) that this is the *exact* degree. The degree in a must fall short of the degree of the denominator by $2n + 1$ exactly; for in the value of $\frac{A}{1 - au}$ we notice that the degree in a of the numerator falls short of that of the denominator by $\frac{(n+\lambda)(n+\lambda+1)}{2}$, which difference remains unaltered, of course, in the subsequent operations; now this difference of degree in a is equivalent to a difference of $\frac{(n+\lambda)(n+\lambda+1)}{2\lambda}$ in a, the least value of which, as λ takes the values $1, 2, \ldots . n$, is attained when $\lambda = n$, and is $2n + 1$: and since this minimum difference is reached *only* when $\lambda = n$, the term involving the corresponding power of a cannot be destroyed. Hence the degree in a of the numerator is $2 + \big(2 + 3 + \ldots . + (2n - 1)\big) + n - (2n + 1) = 2n(n - 1)$.

Secondly, take a quantic of odd order, $2n + 1$,* so that

$$\phi(x) = \frac{1 - x^{-2}}{(1 - ax^{2n+1})(1 - ax^{2n-1}) \ldots . (1 - ax)(1 - ax^{-1}) \ldots . (1 - ax^{-2n+1})(1 - ax^{-2n-1})}$$

and

$$\lambda A = (-)^{\frac{2n+1-\lambda}{2}} \frac{a^{\frac{2n+1-\lambda}{2} \cdot \frac{2n+3-\lambda}{2}}(1 - a^2)}{(1 - a^2)^2(1 - a^4)^2 \ldots . (1 - a^{2n+1-\lambda})^2(1 - a^{2n+3-\lambda}) \ldots . (1 - a^{2n+1+\lambda})},$$

where λ is some odd number between 1 and $2n + 1$, both inclusive. When $\lambda = 2n + 1$, the above denominator (after cancellation of $1 - a^2$) is

$$(1 - a^4)(1 - a^6) \ldots . (1 - a^{4n})(1 - a^2),$$

which is contained in $(1 - a^2)(1 - a^4)(1 - a^6) \ldots . (1 - a^{4n})$;

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* n not zero. What is here and in subsequent parts of the paper given concerning the form of the numerator does not in general apply to the linear quantic.

when λ is less than $2n+1$, the denominator (after cancellation of $1-a^2$) is

$$(1-a^4)(1-a^6)\ldots\ldots(1-a^{2n+1+\lambda})(1-a^2)(1-a^4)\ldots\ldots(1-a^{2n+1-\lambda})$$

which (as can be seen in the same way as the corresponding case for even quantics) is also contained in

$$(1-a^2)(1-a^4)(1-a^6)\ldots\ldots(1-a^{4n}).$$

Hence the partial fractions will have for a common denominator

$$(1-a^2)(1-a^4)(1-a^6)\ldots\ldots(1-a^{4n})(1-ax)(1-ax^3)\ldots\ldots(1-ax^{2n+1}).$$

The degree in x of the numerator is necessarily less by at least 1 than that of the denominator, and the degree in a is seen (in the same manner as for even quantics) to be less by exactly $2n+2$ than that of the denominator. We shall immediately show that the degree in x falls short of the degree of the denominator by exactly 2; hence the degree in x is $(n+1)^2-2$, and the degree in a is $4n^2+n-1$.

The proof that the degree in x of the numerator of the generating function is less by *exactly* 2 than that of the denominator applies to even and odd quantics alike, and is as follows. Denote the generating function obtained by

$$\frac{f(x)}{(1-ax^i)(1-ax^{i-2})\ldots\ldots};$$

this was obtained by adding together those partial fractions which corresponded to the factors in the denominator of $\phi(x)$ in which the exponent of x was positive: hence it is easy to see that

$$\phi(x)=\frac{f(x)}{(1-ax^i)(1-ax^{i-2})\ldots\ldots}-x^{-2}\frac{f(x^{-1})}{(1-ax^{-i})(1-ax^{-(i-1)})\ldots\ldots},$$

i. e. $1-x^{-2}=f(x)(1-ax^{-i})(1-ax^{-(i-2)})\ldots\ldots-x^{-2}f(x^{-1})(1-ax^i)(1-ax^{i-2})\ldots\ldots$

Now the degree in x of

$$x^{-2}f(x^{-1})(1-ax^i)(1-ax^{i-2})\ldots\ldots$$

is less by exactly 2 * than that of

$$(1-ax^i)(1-ax^{i-2})\ldots\ldots;$$

hence, in virtue of the above identity, the degree in x of

$$f(x)(1-ax^{-i})(1-ax^{-(i-2)})\ldots\ldots,$$

* Since it is evident from the mode of formation that $f(x)$, and consequently $f(x^{-1})$, contains a term independent of x.

i. e. of $f(x)$, must also be less by exactly 2 than that of

$$(1-ax^i)(1-ax^{i-2})\ldots\ldots \quad \text{Q. E. D.}$$

The above results may be summed up as follows: That portion of the development of $\phi(x)$ which does not contain negative powers of x can be represented by a fraction, whose denominator for a quantic of order $i=2n$ is

$$(1-a^2)^2(1-a^3)(1-a^4)\ldots\ldots(1-a^{2n-1})(1-ax^2)(1-ax^4)\ldots\ldots(1-ax^{2n})$$

and for a quantic of order $i=2n+1$ is

$$(1-a^2)(1-a^4)(1-a^6)\ldots\ldots(1-a^{4n})(1-ax)(1-ax^3)\ldots\ldots(1-ax^{2n+1}).$$

In both cases, the exponent of the highest power of x that appears in the numerator is less by 2 than the degree in x of the denominator, and the exponent of the highest power of a that appears in the numerator is less by $i+1$ than the degree in a of the denominator.

It is to be observed that we have not proved that the generating function with the denominator above named is in its lowest terms, or even in the lowest terms consistent with the denominator being a product of factors of the forms $1-a^r$, $1-a^r x^s$; nor is it of any special importance whether the fraction is in its lowest terms or not. The difference between the degrees (in a and in x) of the numerator and denominator is of course unaltered by the introduction or suppression of common factors. As a matter of fact, it has been found that in the quantics of the orders 3, 5, 7, 9, 4, 8, a factor $1-a^2$ in the denominator as above given can be suppressed, but that in the quantics of the orders 2, 6, 10, no factor explicitly appearing in the denominator as above given is a divisor of the numerator.

Cayley's Method.

Instead of decomposing the crude generating function

$$\phi(x)=\frac{1-x^{-2}}{(1-ax^i)(1-ax^{i-2})\ldots\ldots(1-ax^{-i+2})(1-ax^{-i})}$$

into partial fractions with respect to x, this method proceeds by decomposing into partial fractions with respect to a. There will be simply $i+1$ such partial fractions, corresponding to the $i+1$ factors of the above denominator, these being linear in a. The fraction corresponding to $1-ax^\lambda$ (where λ is any one of the numbers i, $i-2$, ..., $-i+2$, $-i$) is

$$\frac{1 - x^{-2}}{(1 - x^{i-\lambda})(1 - x^{i-\lambda-2}) \ldots . (1 - x^{2})(1 - x^{-2}) \ldots . (1 - x^{-i-\lambda})} \cdot \frac{1}{1 - ax^{\lambda}}$$

$$= (-1)^{\frac{i+\lambda-2}{2}} \frac{x^{\frac{i+\lambda-2}{2} \cdot \frac{i+\lambda+4}{2}} (1 - x^{2})}{(1 - x^{2})(1 - x^{4}) \ldots . (1 - x^{i-\lambda})(1 - x^{2})(1 - x^{4}) \ldots . (1 - x^{i+\lambda})} \cdot \frac{1}{1 - ax^{\lambda}}.$$

Now this decomposition does not enable us to separate the crude fraction into two parts, such that the expansion of one shall involve *only* negative powers of x and that of the other *no* negative powers of x; for it is plain that the expansion (in ascending powers of a) of every one of the partial fractions of the above form contains an infinite series of positive powers of x. But we now know that if in these expansions we reject the negative powers of x, the sum of the remaining portions is equal to a certain fraction whose denominator is known, and the degree in x and in a of whose numerator is also known. This required numerator is therefore identical with the product of the known denominator by the sum of those portions of the developments of the partial fractions which contain non-negative powers of x; hence we can obtain the numerator by developing each of the partial fractions in ascending powers of a and x, adding the results and multiplying the sum by the known denominator. And since this multiplier contains no negative powers of a or of x, we need go no farther in the expansions of the partial fractions than the known limits of the degrees in a and x of the numerator.

Thus, although the decomposition with respect to a introduces infinite series, we can, in virtue of our knowledge of the form of the result, arrive, by the use of finite portions of these series, at the same generating function as that obtained by Sylvester's method; and the work is in general considerably shorter by the present method. We shall now examine more closely the limits within which the work is confined.

Take, first, a quantic of order $i = 2n$. We have seen that the generating function will have for its denominator

$$(1 - a^{2})^{2}(1 - a^{3})(1 - a^{4}) \ldots . (1 - a^{2n-1})(1 - ax^{2})(1 - ax^{4}) \ldots . (1 - ax^{2n}),$$

and that its numerator will be of the degree $q = n(n + 1) - 2$ in x and of the degree $p = 2n(n - 1)$ in a. Let μ be any one of the positive integers $1, 2, \ldots . n$. The partial fractions corresponding to $1 - ax^{2\mu}$, $1 - a$, $1 - ax^{-2\mu}$, respectively, are

$$(-1)^{n+\mu-1} \frac{x^{(n+\mu-1)(n+\mu+2)}(1 - x^{2})}{(1 - x^{2})(1 - x^{4}) \ldots . (1 - x^{2n-2\mu})(1 - x^{2})(1 - x^{4}) \ldots . (1 - x^{2n+2\mu})} \cdot \frac{1}{1 - ax^{2\mu}},$$

$$(-1)^{n-1} \frac{x^{(n-1)(n+2)}(1 - x^{2})}{(1 - x^{2})(1 - x^{4}) \ldots . (1 - x^{2n})(1 - x^{2})(1 - x^{4}) \ldots . (1 - x^{2n})} \cdot \frac{1}{1 - a},$$

$$(-1)^{n-\mu-1}\frac{x^{(n-\mu-1)(n-\mu+2)}(1-x^2)}{(1-x^2)(1-x^4)\ldots\ldots(1-x^{2n+2\mu})(1-x^2)(1-x^4)\ldots\ldots(1-x^{2n-2\mu})}\cdot\frac{1}{1-ax^{-2\mu}}.$$

In the expansion of the first of the above expressions, the lowest exponent of x is $(n+\mu-1)(n+\mu+2)$, which is greater than q; hence we reject all the partial fractions corresponding to those factors of the denominator of the crude fraction in which the exponent of x is positive. The expansion of the first factor of the second expression is

$$(-1)^{n-1}\, x^{(n-1)(n+2)} + \text{terms of higher degree in } x;$$

accordingly, since $(n-1)(n+2)=q$, the portion of the expansion of the partial fraction corresponding to $1-a$ which we have to take into account is

$$(-1)^{n-1}\, x^q\,(1+a+a^2+\ldots\ldots+a^p).$$

With regard to the third expression (the partial fraction corresponding to $1-ax^{-2\mu}$), we observe that the needed portion of the expansion of its second factor is

$$1+ax^{-2\mu}+a^2x^{-4\mu}+\ldots\ldots+a^px^{-2p\mu},$$

so that we must expand the first factor as far as the power $x^{q+2p\mu}$, and multiply the result by the above series, rejecting in the product negative powers of x and also positive powers higher than q. Finally, we multiply the sum of the retained portions of all these expansions by the denominator

$$(1-a^2)^2(1-a^3)(1-a^4)\ldots\ldots(1-a^{2n-1})(1-ax^2)(1-ax^4)\ldots\ldots(1-ax^{2n});$$

retaining in the product only powers of a not higher than p and powers of x not higher than q, we have the required numerator.

For a quantic of odd order, $i=2n+1$, we find in like manner that we have to take account only of the partial fractions corresponding to those factors of the denominator of the crude fraction in which the exponent of x is negative. The denominator of the required generating function is

$$(1-a^2)(1-a^4)(1-a^6)\ldots\ldots(1-a^{4n})(1-ax)(1-ax^3)\ldots\ldots(1-ax^{2n+1});$$

the degrees of the numerator in a and x are $p=4n^2+n-1$, $q=(n+1)^2-2$. The partial fraction corresponding to $1-ax^{-2\mu-1}$ is

$$(-1)^{n-\mu-1}\frac{x^{(n-\mu-1)(n-\mu+2)}(1-x^2)}{(1-x^2)(1-x^4)\ldots\ldots(1-x^{2n+2\mu+2})(1-x^2)(1-x^4)\ldots\ldots(1-x^{2n-2\mu})}\cdot\frac{1}{1-ax^{-2\mu-1}};$$

we expand the first factor as far as the power $x^{q+(2\mu+1)p}$, and multiply the result by

$$1+ax^{-2\mu-1}+a^2x^{-4\mu-2}+\ldots\ldots+a^px^{-2p\mu-p}.$$

Collecting the terms thus obtained for $\mu = 0, 1, 2, \ldots, n$, we multiply the sum by the denominator above given; the terms of the product within the assigned limits of degree constitute the required numerator.

In practice, we calculate the numerator only as far as the power $\frac{p}{2}$ or $\frac{p+1}{2}$ (say p') of a; a known symmetry enabling us to write the remaining half without calculation. The extent to which each of the expansions has to be carried is then given by putting p' for p in the preceding formulæ; and it is obvious that the abridgment of the work is very much more than half, even without taking account of the fact that the numerical coefficients in the expansions rapidly increase in magnitude. — The same symmetry would enable us to dispense with the calculation of half the powers of x instead of a, but it is plain that this would result in much less saving.

The symmetry referred to is proved as follows: —

Let the generating function for a quantic of order $2n + 1$ be

$$F(x) = \frac{A_0 + A_1 x + \ldots + A_q x^q}{(1 - ax)(1 - ax^3)\ldots(1 - ax^{2n+1})(1 - a^\alpha)(1 - a^\beta)\ldots},$$

and as before let p denote the highest power of a in the numerator. Then we have

$$\phi(x) = F(x) - x^{-2}F(-x)$$

i. e.

$$\begin{aligned}(1-x^{-2})(1-a^\alpha)(1-a^\beta)\ldots = {} & (A_0 + A_1 x + \ldots + A_q x^q)(1 - ax^{-1})(1 - ax^{-3})\ldots(1 - ax^{-2n-1}) \\ & - x^{-2}(A_0 + A_1 x^{-1} + \ldots + A_q x^{-q})(1 - ax)(1 - ax^3)\ldots(1 - ax^{2n+1}).\end{aligned}$$

It is to be noticed (see p. 142, l. 9) that $\alpha + \beta + \ldots = p + n + 1$, and that $1 + 3 + \ldots + (2n + 1) = q + 2$; let us denote the number of the factors $1 - a^\alpha, 1 - a^\beta, \ldots$ by k. In the above identity, replace a by a^{-1} and x by x^{-1}, and multiply both sides by $(-1)^{n+1}a^{p+n+1}x^{-2}$; then, denoting by A'_μ what A_μ becomes when in it a is changed into a^{-1} and the result multiplied by a^p, the identity becomes

$$\begin{aligned}(-1)^{n-k}(1-x^{-2})(1-a^\alpha)(1-a^\beta)\ldots = {} & (A'_0 x^q + A'_1 x^{q-1} + \ldots + A'_q)(1-ax^{-1})(1-ax^{-3})\ldots(1-ax^{-2n-1}) \\ & - x^{-2}(A'_0 x^{-q} + A'_1 x^{-q+1} + \ldots + A'_q)(1-ax)(1-ax^3)\ldots(1-ax^{2n+1}).\end{aligned}$$

Comparing this with the former identity, we see that

$$A'_\mu = (-1)^{n-k}A_{q-\mu},$$

or, in other words, the coefficients of $a^\lambda x^\mu$ and $a^{p-\lambda}x^{q-\mu}$ in the numerator of $F(x)$ are equal in absolute value, and have like or opposite signs throughout, according as $n - k$ is even or odd.

In like manner, we have for a quantic of order $2n$

$$F(x)=\frac{A_0+A_2x^2+\ldots.+A_qx^q}{(1-ax^2)\,(1-ax^4)\ldots.(1-ax^{2n})\,(1-a^\alpha)\,(1-a^\beta)\ldots.},$$

whence

$$\begin{aligned}(1-x^{-2})\,(1-a^\alpha)\,(1-a^\beta)\ldots.=&(A_0+A_2x^2+\ldots.+A_qx^q)\,(1-a)\,(1-ax^{-2})\,(1-ax^{-4})\ldots.(1-ax^{-2n})\\&-x^{-2}(A_0+A_2x^{-2}+\ldots.+A_qx^{-q})\,(1-a)\,(1-ax^2)\,(1-ax^4)\ldots.(1-ax^{2n}).\end{aligned}$$

Denoting again by A'_μ what A_μ becomes when a is changed into a^{-1} and the result multiplied by a^p, we derive as before

$$\begin{aligned}(-1)^{n-k}(1-x^{-2})(1-a^\alpha)(1-a^\beta)\ldots.=&(A'_0x^q+A'_2x^{q-2}+\ldots.+A'_q)(1-a)(1-ax^{-2})(1-ax^{-4})\ldots.(1-ax^{-2n})\\&-x^{-2}(A'_0x^{-q}+A'_2x^{-q+2}+\ldots.+A'_q)(1-a)(1-ax^2)(1-ax^4)\ldots.(1-ax^{2n}).\end{aligned}$$

By comparing these identities we find

$$A'_\mu=(-1)^{n-k}A_{q-\mu};$$

i. e. the coefficients of $a^\lambda x^\mu$ and $a^{p-\lambda}x^{q-\mu}$ in the numerator of $F(x)$ are equal in absolute value, and of like or opposite signs according as $n-k$ is even or odd.

Professor Cayley has given a full statement of the calculation in the case of the seventhic in this Journal (Vol. II. pp. 71–84); he there mentions (p. 75) a question (which I have passed over) concerning the legitimacy of the mode adopted in expanding the partial fractions.

Another point of symmetry may be deduced from the above identities. For even quantics, comparing only the coefficients of the highest negative power of x on the two sides of the identity, we find

$$(-1)^n a^n A_0 - A_q = 0;$$

but we have seen that $A_q=(-1)^{n-k}A'_0$; therefore

$$A_0=(-1)^k a^{-n}A'_0;$$

i. e. the invariantive part of the numerator has a symmetry of its own, the coefficient of a^λ being $(-1)^k$ times the coefficient of $a^{p-n-\lambda}$; of course a corresponding symmetry holds for A_q. For odd quantics we find in like manner that the coefficient of any term a^λ in A_0 is $(-1)^{k-1}$ times the coefficient of $a^{p-n-1-\lambda}$ in the same.

Sylvester's Second Method.

This method, like Professor Cayley's, is based upon the fact that we know *à priori* the denominator and the degrees of the numerator of the generating function. It was devised by Professor Sylvester shortly after Professor Cayley had invented the one just explained, and before the latter had made it known.

The process is extremely simple. If the generating function for a quantic of the order i were expanded, the coefficient of any term $a^j x^g$ in the expansion would be $\Delta(i, j : g)$. The coefficients of the expansion can therefore be obtained by formula (5), p. 129. Now the expansion multiplied by the known denominator of the generating function gives the numerator; and since the latter is confined to known limits of degree in a and in x (while the denominator contains no negative powers of a or x), we have to use only those terms of the expansion which fall within these limits and multiply their sum by the known denominator; that portion of the product which falls within the limits of degree is the required numerator.

The work, then, can be stated as follows. Let the order of the quantic be i, the denominator of the required generating function D, and the degrees in a and x of the numerator p and q. Develop the fractions of the form

$$\frac{(1-z^{j+1})(1-z^{j+2})\ldots\ldots(1-z^{j+i})}{(1-z^2)(1-z^3)\ldots\ldots(1-z^i)}$$

obtained by giving to j all values from $j=0$ to $j=p$. The coefficient of z^w in the development of the above fraction is $\Delta(w : i, j) = \Delta(i, j : ij - 2w)$ and is therefore the coefficient of $a^j x^{ij-2w}$ in the expansion of the generating function. If, then, ij is even, the coefficients of $z^{\frac{ij}{2}}$ and of the $\frac{1}{2}q$ (or $\frac{1}{2}(q-1)$) next preceding terms in the development of the above fraction are the coefficients of $a^j x^0, a^j x^2, \ldots, a^j x^q$ (or $a^j x^{q-1}$) in the expansion of the generating function; if ij is odd, the coefficients of $z^{\frac{ij-1}{2}}$ and of the $\frac{1}{2}(q-1)$ (or $\frac{1}{2}(q-2)$) next preceding terms are the coefficients of $a^j x^1, a^j x^3, \ldots, a^j x^q$ (or $a^j x^{q-1}$) in the expansion of the generating function. Multiplying the aggregate of the terms thus obtained by D, restricting the product to terms whose degrees do not surpass p, q we have the required numerator. Of course, we do not actually use values of j higher than p' $(=\frac{1}{2}p$ or $\frac{1}{2}(p+1))$, the remaining half of the numerator being obtained by symmetry.

As none of the larger calculations have been performed by more than one of the methods described, I cannot say positively whether this or Professor Cayley's method is practically the more desirable, but I believe that this method is to be preferred. It has the advantage of greater uniformity and less liability to accidental errors, and there is, I think, no very great difference (on which side I do not know) in the amount of numerical work required.

It may be well to mention at this place the great advantage of employing paper ruled into squares, in performing the calculations required by any of these

methods. For instance, in the *developments* required by the method last described, the only operations that occur are those of multiplication and division by factors of the form $1 - z^\lambda$. The coefficients of the successive powers of z are written in the successive squares of a row: to multiply by $1 - z^\lambda$, the same series of coefficients shoved λ places forward is subtracted from the given series; to divide by $1 - z^\lambda$, the first λ terms of the given series are written down as the first λ terms of the quotient, and from this point on, the series of quotient-coefficients is continued by being shoved λ places forward and added to the given series. I find it convenient not to write down the figures to be subtracted or added; multiplication by $1 - z^\lambda$ being performed by subtracting from each coefficient of the multiplicand the coefficient λ spaces back of it, and division by adding to each coefficient of the dividend the coefficient λ spaces back of it in the quotient: the regularity of the squares enables one to do this without danger of error. The advantage of this use of paper ruled into squares is even greater when powers of *two* letters (say a and x) are involved. The coefficients are then arranged in horizontal and vertical rows of squares according to the powers of a and x (as in the tables, this Journal, Vol. II. pp. 226–246); and, e. g., multiplication by $1 - ax^3$ is performed by subtracting from each coefficient of the multiplicand the coefficient one space above and three spaces to the left of it. It is needless to go into further details on this point; a very great amount of time is saved by this practically almost indispensable contrivance, which any one who had occasion for it would know how to use to the best advantage.

It may be worth while to say a few words concerning the arrangement of the work (as actually performed) in Sylvester's second method. We first develop the fraction

$$\frac{(1 - z^{p'+1})(1 - z^{p'+2}) \ldots . (1 - z^{p'+i})}{(1 - z^2)(1 - z^4) \ldots . (1 - z^i)}$$

(where p' is the highest value of j required) as far as it is needed; this result divided by $1 - z^{p'+i}$ and multiplied by $1 - z^{p'}$ gives the series corresponding to $j = p' - 1$; this divided by $1 - z^{p'+i-1}$ and multiplied by $1 - z^{p'-1}$ gives the series corresponding to $j = p' - 2$; we continue in this way, obtaining the series for each value of j from the next higher, till we reach $j = 0$, for which the value of the fraction is $1 - z$: and it is easy to see that (if each series is carried a few terms beyond the point actually needed) we have thus a perfect test of this portion of the work. In other parts of the work, obvious and easy partial tests can be made; and a very good test of the correctness of the result obtained for the numerator of the generating function is afforded by the symmetry of A_0 (see p. 146). For in the calculation of A_q all or nearly all the

coefficients in $A_0, A_1, A_2, \ldots . A_{q-1}$ are involved; now it is by means of A_q that (from the rule of symmetry) we complete A_0, and if this completion makes A_0 symmetrical, A_q and A_0 are perfectly tested, and consequently the intervening A's pretty thoroughly. Of course, if instead of calculating half the numerator we calculated the whole, its symmetry would be a perfect test, and no other test of the result would be needed; but in the longer cases this would multiply the labor many times. There are many short and satisfactory ways of checking the work in its progress, which would suggest themselves to any calculator, and which it would be tedious to detail. Another test of the *result*, which with the one above given puts its correctness beyond the region of practical doubt, will be mentioned in connection with the generating functions for differentiants.

Generating Functions for Differentiants.

If we put $x = 1$ in the expansion of the generating function, the coefficient of a^j will be the number of linearly independent covariants of the degree j and of all orders; or, say, the number of linearly independent *differentiants* of the degree j (and of all weights). Hence by putting $x = 1$ in the result given by any of the methods above described, we obtain what Professor Sylvester calls the *generating function for differentiants.*

But this generating function can also be obtained more directly. The number of linearly independent differentiants of the degree j and of all weights is

$$\Delta(W:i,j)+\Delta(W-1:i,j)+\Delta(W-2:i,j)+\ldots.+\Delta(2:i,j)+\Delta(1:i,j)+\Delta(0:i,j),$$

where W is the highest weight which a differentiant of degree j can have, viz. $\frac{ij}{2}$ or $\frac{ij-1}{2}$; and it is easily seen that $\Delta(0:i,j)$ is to be regarded as 1. Writing this sum more explicitly

$$(W:i,j)-(W-1:i,j)+(W-1:i,j)-(W-2:i,j)+\ldots.+(2:i,j)-(1:i,j)+(1:i,j)-(0:i,j)+1$$

we see that its value is $(W:i,j)$. Now $(W:i,j)$ is the coefficient of $a^j x^{ij-2W}$ in the development of

$$\frac{1}{(1-ax^i)(1-ax^{i-2})\ldots.(1-ax^{-i+2})(1-ax^{-i})}.$$

If i is even, $ij-2W$ is 0 for all values of j; if i is odd, $ij-2W$ is 0 or 1 according as j is even or odd. Accordingly, if i is even, the multiplier of x^0, and if i is odd, the sum of the multipliers of x^0 and x^1,* in the development of the above fraction, is the generating function for differentiants.

* When i is odd, even powers of x are multiplied only by even powers of a and odd powers only by odd powers, in the development; so that in this addition each power of a is taken only once, as it should be.

This generating function is calculated by means of the decomposition of the above fraction into partial fractions with respect to x, as in "Sylvester's First Method."

Suppose $i = 2n$. The fraction corresponding to $1 - ax^{2\lambda}$ (λ positive) is

$$\Sigma \frac{A}{1 - \rho a^{\frac{1}{\lambda}} x^2} = \Sigma \frac{A}{1 - ax^2}, \text{ say;}$$

where

$$\lambda A = (-)^{n-\lambda} \frac{a^{\frac{(n-\lambda)(n-\lambda+1)}{2}}}{(1-a)^2 (1-a^2)^2 (1-a^3)^2 \ldots\ldots (1-a^{n-\lambda})^2 (1-a^{n-\lambda+1}) \ldots\ldots (1-a^{n+\lambda})}.$$

In the development of this fraction, the multiplier of x^0 is simply ΣA; so that the generating function for differentiants will be obtained by adding together ΣA and the analogous expressions for all the factors of the denominator in which the exponent of x is positive. It is plain, from what has been given at p. 139, that these expressions will have for a common denominator

$$(1-a)(1-a^2)^2(1-a^3)(1-a^4) \ldots\ldots (1-a^{2n-1}).$$

The degree of the numerator is (see p. 140) less by $2n + 1$ than that of the denominator.

Suppose $i = 2n + 1$. The fraction corresponding to $1 - ax^{\lambda}$ (λ positive) is

$$\Sigma \frac{A}{1 - \rho a^{\frac{1}{\lambda}} x} = \Sigma \frac{A}{1 - ax}, \text{ say;}$$

where

$$\lambda A = (-)^{\frac{2n+1-\lambda}{2}} \frac{a^{\frac{2n+1-\lambda}{2} \cdot \frac{2n+3-\lambda}{2}}}{(1-a^2)^2(1-a^4)^2 \ldots\ldots (1-a^{2n+1-\lambda})^2 (1-a^{2n+3-\lambda}) \ldots\ldots (1-a^{2n+1+\lambda})}.$$

In the development of this fraction, the sum of the multipliers of x^0 and x^1 is $\Sigma A(1+a)$; and the generating function for differentiants is obtained by adding together $\Sigma A(1+a)$ and the similar expressions corresponding to the several factors of the denominator in which the exponent of x is positive. These expressions will have for a common denominator (see p. 140)

$$(1-a)(1-a^2)(1-a^4)(1-a^6) \ldots\ldots (1-a^{4n}).$$

The degree of the numerator is (see p. 141) less by $2n + 2$ than that of the denominator.

Thus the calculation of the generating function for differentiants is much shorter than that of the generating function for covariants; and the value of the former, thus independently obtained, affords a perfect test of the correctness of the latter: putting $x = 1$ in the generating function for covariants, the resulting fraction must be identical in value with the generating function for differentiants,

with which it can be compared with very little labor by reduction to a common denominator.—But for practical certainty it is not necessary to make an independent calculation of the generating function for differentiants at all; our knowledge of its *denominator* suffices to give a very searching test. The denominator of the generating function for covariants (in all but the lowest orders of quantics) becomes, on putting $x = 1$, the product of the denominator of the generating function for differentiants by a power of $1 - a$; the numerator of the generating function for covariants must therefore, on putting $x = 1$, become divisible by this power of $1 - a$; the performance of this division therefore at once very thoroughly tests the generating function for covariants and gives us the generating function for differentiants by a self-checking process, which renders the independent calculation of the latter unnecessary.

It seems needless to set forth two other methods which might be used for calculating the generating function for differentiants, having the same relation to the above as Cayley's method and Sylvester's second method of calculating the generating function for covariants have to Sylvester's first method.

Systems of Quantics.

The methods of obtaining the generating functions and groundforms for systems of quantics are extensions of the methods used for single quantics; it seems unnecessary to set them out at length, a very brief account being now sufficient to make them intelligible.

We must first define an extension of the notation $(w : i, j)$. The number of ways in which w can be composed by the addition of j numbers taken from the set $0, 1, 2, \ldots . i$, together with j' numbers from the set $0, 1, 2, \ldots . i'$, j'' from the set $0, 1, 2, \ldots . i''$, etc. is denoted by the symbol

$$(w : i, j; i', j'; i'', j''; \ldots .)$$

E. g., $(5 : 3, 4; 4, 2) = 21$, since 5 can be composed in the following 21 ways, using four of the numbers $0, 1, 2, 3$ and two of the numbers $0, 1, 2, 3, 4$:—

3200,00	3110,00	2210,00	2111,00	3100,10	2200,10
2110,10	1111,10	2100,20	2100,11	1110,20	1110,11
2000,30	2000,21	1100,30	1100,21	1000,40	1000,31
1000,22	0000,41	0000,32*			

* It is obvious that $(w : i, j; i', j'; i'', j''; \ldots .) = \Sigma\Sigma\Sigma \ldots . (v : i, j)(v' : i', j')(v'' : i'', j'') \ldots .$; the summations referring to the v's, which are to take all positive integer values (including 0) subject to the condition $v + v' + v'' + \ldots . = w$.

The fundamental theorem on which the investigation of the generating functions and groundforms of a system of quantics rests is an extension of the fundamental theorem for a single quantic, and was demonstrated by Professor Sylvester along with the latter. It is as follows: —

The number of linearly independent differentiants of weight w and degrees $j, j', j''\ldots.$ in the coefficients of a system of quantics of the orders $i, i', i''\ldots.$ respectively is $(w: i, j; i', j'; i'', j''; \ldots.) - (w-1: i, j; i', j'; i'', j''; \ldots.)$.

It follows, in the same manner as the corresponding theorem for single quantics, that the number of linearly independent covariants of order g and degrees $j, j', j''\ldots.$ is the coefficient of $a^j b^{j'} c^{j''}\ldots. x^g$ in the development of

$$\frac{1-x^{-2}}{(1-ax^i)(1-ax^{i-2})\ldots.(1-ax^{-i})(1-bx^{i'})(1-bx^{i'-2})\ldots.(1-bx^{-i'})(1-cx^{i''})(1-cx^{i''-2})\ldots.(1-cx^{-i''})\ldots.}$$

in ascending powers of $a, b, c\ldots.$

If this fraction be decomposed into partial fractions with respect to x as in "Sylvester's First Method," we find, as in the case of a single quantic, that the partial fractions corresponding to those factors of the denominator in which the exponent of x is positive, are the only ones whose development in ascending powers of $a, b, c\ldots.$ gives non-negative powers of x; so that the generating function for the covariants of a system of quantics is obtained by adding together *these* partial fractions only. The process is thus an obvious extension of "Sylvester's First Method," which need not be further explained.

There is a feature of the calculation, however, that does not appear in the case of a single quantic. This will be sufficiently shown by an example; let us take the system of a quadric and a cubic. The crude form of the generating function is

$$\frac{1-x^{-2}}{(1-bx^2)(1-b)(1-bx^{-2})(1-cx^3)(1-cx)(1-cx^{-1})(1-cx^{-3})}.$$

The sum of the partial fractions corresponding to $1-bx^2$ has for its denominator

$$(1-b)(1-bc^2)(1-b^3c^2)(b-c^2)(b^3-c^2);$$

the like sums corresponding to $1-cx^3$ and $1-cx$ have for their denominators respectively

$$(1-b)(1-c^2)(1-c^4)(1-b^3c^2)(b^3-c^2),$$

and

$$(1-b)(1-c^2)(1-c^4)(1-bc^2)(b-c^2).$$

The sum of all these fractions, which is the generating function sought, would therefore appear to contain in its denominator the factors $b-c^2$ and b^3-c^2; and it is plain that the presence of these factors in the denominator would render the expansion of the fraction in simultaneously ascending powers of b and c

impossible. But as a matter of fact, the *numerator* obtained by adding the partial fractions also contains $b - c^2$ and $b^3 - c^2$ as factors; and on cancellation of them, the generating function has essentially the same character as that of a single quantic. The feature above exemplified appears in all cases of systems of quantics.

All the generating functions for systems of quantics that have as yet been obtained were calculated by the method above indicated. It is easy to see the course that would have to be pursued in basing upon it extensions of Cayley's method and of Sylvester's second method.

When the reduced form of the generating function has been obtained, it is converted into the representative form in a mode analogous to that employed for a single quantic.

The table of groundforms is deduced from the representative generating function by precisely the same process as for a single quantic. The validity of the process is dependent on the same postulate, and, granting that postulate, proved by the same reasoning as in the case of a single quantic. In fact, if we merely change *degorder* $(j \, . \, g)$ into *degorder* $(j, j', j'' \, \ldots\ldots \, g)$ and $a^r x^s$ into $a^r b^t c^u \ldots . \, x^s$ throughout, every word of the argument under the head "Mode of obtaining the Table of Groundforms" applies to a system of any number of quantics.

Finally, it may be mentioned that the generating function for differentiants may be obtained from that for covariants by putting $x = 1$ in the latter. It may also be obtained independently (and in general in lower terms) by a process precisely analogous to that explained under the head "Generating Functions for Differentiants."

Notes on Modern Algebra.

By Cav. Faà de Bruno, *Turin.*

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1. *Application du Théorème donné par l'Auteur sur le Développement des Fonctions.*

Le théorème que j'ai déjà donné en 1875, comme on peut voir dans mon ouvrage (Théorie des Formes Binaires, p. 311), précité qui m'a été revendiquée dans ce journal même par M. Sylvester (voir année 1879, p. 351) nous permet de découvrir une belle propriété des covariants.

D'abord, si ϕ est un covariant d'une forme donnée $f=(a_0, a_1 \ldots . a_n)(x, y)^n$, et si nous posons,

$$\delta = a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + 3a_2 \frac{d}{da_3} + \ldots , \tag{1}$$

il est aisé de voir qu'en appellant c_m le dernier coefficient de ϕ, supposé d'ordre m, on aura cette nouvelle propriété que

$$\phi = e^{\delta x} c_m. \tag{2}$$

On y arrive facilement en ayant égard à la propriété qui ont les coefficients de se déduire du premier ou du dernier par des différentiations successives (voir la susdite "Théorie," p. 188).

Or par le théorème cité, il s'ensuit que, puisque $\phi = e^{\delta x} c_m$, on obtiendra ϕ en développant c_m lorsqu'on y remploie :

$$\begin{array}{lll} a_0 & \text{par} & A_0 = a_0, \\ a_1 & \text{"} & A_1 = a_1 + a_0 x, \\ a_2 & \text{"} & A_2 = a_2 + 2a_1 x + a_0 x^2, \\ a_3 & \text{"} & A_3 = a_3 + 3a_2 x + 3a_1 x^2 + a_0 x^3. \end{array}$$

On a donc ce théorème :

Tout covariant est le développement d'un différentiant où au lieu des a on met les A correspondants.

EXEMPLES. **Prenons** ainsi la cubique

$$a_0 x_3 + 3\,a_1 x^2 y + 3\,a_2 xy^2 + a_3 y^3\,;$$

le covariant quadratique sera le développement de $a_1 a_3 - a_2^2$ transformé en $A_1 A_3 - A_2^2$. On aura ainsi ($y = 1$),

$$(a_0 a_2 - a_1^2)\,x^2 + (a_0 a_3 - a_1 a_2)\,x + a_1 a_3 - a_2^2 = (a_1 + a_0 x)\,(a_3 + 3\,a_2 x + 3\,a_1 x^2 + a_0 x_3) \\ - (a_2 + 2\,a_1 x + a_0 x^2)^2.$$

On aura aussi pour le covariant quadratique de la quintique, en négligeant les termes superflus en x^3, x^4,

$$(a_0 a_4 - 4\,a_1 a_3 + 3\,a_2^2)\,x^2 + (a_0 a_5 - 3\,a_1 a_4 + 2\,a_2 a_3)\,x + a_1 a_5 - 4\,a_2 a_4 + 3\,a_2^2 \\ = (a_1 + a_0 x)\,(a_5 + 5\,a_4 x + 10\,a_3 x^2) - 4\,(a_2 + 2\,a_1 x + ax_2)\,(a_4 + 4\,a_3 x + 6\,a_2 x^2 \\ + 3\,(a_3 + 3\,a_2 x + 3\,a_1 x^2)^2.$$

Ajoutons encore que comme (v. "Théorie," p. 130) nous avons démontré que $A_i = e^\delta A_i$, et que par conséquent $A_{i-1} = \frac{1}{i}\,\delta - Ai$, et ainsi, de suite, on conclura en vertu de l'équation (2) que tout covariant est une *fonction symbolique de* δ *appliquée au dernier terme* a_n. Ainsi pour tout covariant quadratique ϕ il viendra

$$\phi = \frac{1}{n\,(n-1)}\, e^{\delta x} a_n \,.\, e^{\delta x} \delta^2 a_n - \frac{1}{n^2}\,(e^{\delta x}, \delta\, a_n)^2.$$

La fonction même f sera

$$f = e^{\delta x} a_n.$$

2. *Sur un Théorème dans les Déterminants.*

Soit un déterminant

$$D = \begin{vmatrix} A_1 & \Delta A_1 & \Delta^2 A_1 \,\ldots\ldots & \Delta^n A \\ B & \Delta B_1 & \Delta^2 B_1 \,\ldots\ldots & \Delta^n B \\ \cdot & \cdot & \cdot \quad \cdot & \cdot \end{vmatrix}$$

tel que chaque colonne soit élément par élément déduite de celle qui précède par l'opération quelconque Δ, on aura

$$\Delta D = 0$$

si la dernière colonne est une suite de constantes par rapport à Δ.

En effet par un principe connu sur la différentiation des déterminants, le résultat se réduira au déterminant D où la dernière colonne sera opérée par Δ, et par l'hypothèse admise, tous les éléments seront 0.

Ce théorème peut faciliter beaucoup de recherches et de démonstrations.

EXEMPLES. Soit le déterminant

$$D = \begin{vmatrix} xyz & xy+yz+zx & x+y+z & 1 \\ yzt & yz+zt+ty & y+z+t & 1 \\ ztx & zt+tx+xz & z+t+x & 1 \\ txy & tx+xy+yt & t+x+y & 1 \end{vmatrix}$$

et posons
$$\Delta = \frac{d}{dx} + \frac{d}{dy} + \frac{d}{dz} + \frac{d}{dt},$$

il viendra
$$\Delta D = 0,$$

et par conséquent par un théorème connu (voir notre "Théorie des Formes Binaires"), D sera le produit des différences $x-y$, $y-z$, etc.

$$D = (x-y)(y-z)(z-t)(t-x)(x-z)(y-t).$$

On trouverait par le même procédé ($\Delta = \delta_x + \delta_y + \delta_z + \delta_t$)

$$D' = \begin{vmatrix} (x+y+z)^3, (x+y+z)^2, x+y+z, 1 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{vmatrix}$$

$D' = D^2$, D étant le produit ci-dessus.

EXEMPLE 2^e. Soit

$$D = \begin{vmatrix} p^m, p^m p', \ldots . p'^m \\ p^{m-1} q \qquad p'^{m-1} q' \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ q^m \quad q^{m-1} q' \ldots . q'^m \end{vmatrix} {}^*$$

Posons
$$\Delta = p' \frac{d}{dp} + q' \frac{d}{dq},$$
$$\Delta' = p \frac{d}{dp'} + q \frac{d}{dq'},$$
$$\Delta'' = p \frac{d}{dq'} + p' \frac{d}{dq'},$$
$$\Delta''' = q \frac{d}{dp} + q' \frac{d}{dp'}.$$

On pourra obtenir le déterminant D, en opérant à volonté par Δ, ou par Δ', Δ'',

* Ce déterminant est le même qu'on retrouve dans le *catalecticant* quand on veut démontrer qu'il est un covariant (voir l'ouvrage ci-dessus, p. 726).

Δ''', sur la première colonne, ou sur la dernière colonne, sur la première ou sur la dernière ligne. De sorte qu'on aura par le théorème ci-dessus,

$$p'\frac{dD}{dp}+q'\frac{dD}{dq}=0,$$

$$p\frac{dD}{dp'}+q\frac{dD}{dq'}=0,$$

$$p\frac{dD}{dq}+p'\frac{dD}{dq'}=0,$$

$$q\frac{dD}{dp}+q'\frac{dD}{dp'}=0.$$

Ajoutons que comme D est une fonction évidemment homogène en p, q, p', q', on a encore

$$p\frac{dD}{dp}+p'\frac{dD}{dp'}+q\frac{dD}{dq}+q'\frac{dD}{dq'}=2\mu D,$$

en appellant 2μ le degré de D.

On tire des 4 équations ci-dessus,

$$p\frac{dD}{dp}-q'\frac{dD}{dq'}=0\ ,\ p'\frac{dD}{dp'}-q\frac{dD}{dq}=0,$$

d'où

$$p\frac{dD}{dp}+p'\frac{dD}{dp'}-q\frac{dD}{dq}-q'\frac{dD}{dq'}=0.$$

En la combinant avec celle d'Euler, il vient

$$p\frac{dD}{dp}+p'\frac{dD}{dp'}=\mu D,$$

$$q\frac{dD}{dq}+q'\frac{dD}{dq'}=\mu D.$$

Or les systèmes

$$\left\{\begin{array}{l} p\dfrac{dD}{dp}+p'\dfrac{dD}{dp'}=\mu D \\ q\dfrac{dD}{dp}+q'\dfrac{dD}{dp'}=0 \end{array}\right. , \quad \left\{\begin{array}{l} q\dfrac{dD}{dq}+q'\dfrac{dD}{dq'}=\mu D \\ p\dfrac{dD}{dq}+p'\dfrac{dD}{dq'}=0 \end{array}\right.$$

conduiront par intégration, tous les deux indifféremment à l'équation

$$d\log D=\mu d\log(pq'-p'q),$$

d'où

$$D=(pq'-p'q^{\mu})a,$$

a étant une constante, qu'il est aisé de voir $=1$. Ainsi il serait démontré très facilement que le déterminant D est une puissance simplement du binôme $pq'-p'q$.

EXEMPLE 3ᵉ. Soit

$$D = \begin{vmatrix} \log^3 x & \log x & 1 \\ \log^2 y & \log y & 1 \\ \log z & \log z & 1 \end{vmatrix}$$

et $\Delta = x\dfrac{d}{dx} + y\dfrac{d}{dy} + z\dfrac{d}{dz}$; il s'en rendra que D satisfait à cette équation

$$x\frac{dD}{dx} + y\frac{dD}{dy} + z\frac{dD}{dz} = 0.$$

3. *Sur une Propriété du Jacobien.*

M. Clebsch donne dans sa "Théorie des Formes Binaires," page 119, un théorème sur le carré d'un Jacobien, qu'il démontre à l'aide de ses notations symboliques. Comme je crois qu'il est utile pour le progrès de l'analyse que l'on puisse se passer autant que l'on peut de ces notations, afin d'habituer les élèves à attaquer directement les questions par l'analyse ordinaire, j'ai pensé de donner ici une démonstration de ce théorème, que l'on peut énoncer ainsi. *Le carré du Jacobien de deux formes* donnés est une fonction *quadratique homogène de ces mêmes formes.*

Soient ϕ, ψ les deux formes données de degré m, n, respectivement. Posons

$$\begin{vmatrix} \phi'_x & \phi'_y \\ \psi'_x & \psi'_y \end{vmatrix} = I. \qquad (1)$$

En rappellant l'équation d'Euler $x\phi'_x + y\phi'_y = m\phi$, l'équation (1) pourra s'énoncer ainsi :

$$\begin{vmatrix} x\phi''_x + y\phi''_{xy} + \phi'_x \;, & x\phi''_{xy} + y\phi''_y + \phi'_y \\ x\psi''_x + y\psi''_{xy} + \psi'_x \;, & x\psi''_{xy} + y\psi''_y + \psi'_y \end{vmatrix} = mn\,I.$$

En développant le déterminant, et en observant que

$$x\left\{\begin{vmatrix} \phi''_x & \phi'_y \\ \psi''_x & \psi'_y \end{vmatrix} + \begin{vmatrix} \phi'_x & \phi''_{xy} \\ \psi'_x & \psi''_{xy} \end{vmatrix}\right\} + y\left\{\begin{vmatrix} \phi''_{xy} & \phi'_y \\ \psi''_{xy} & \psi'_y \end{vmatrix} + \begin{vmatrix} \phi'_x & \phi''_y \\ \psi'_x & \psi''_y \end{vmatrix}\right\} = (m + n - z)\,I,$$

il viendra

$$x^2\begin{vmatrix} \phi''_x & \phi''_{xy} \\ \psi''_x & \psi''_{xy} \end{vmatrix} + xy\begin{vmatrix} \phi''_x & \phi''_y \\ \psi''_x & \psi''_y \end{vmatrix} + y^2\begin{vmatrix} \phi''_{xy} & \phi''_y \\ \psi''_{xy} & \psi''_y \end{vmatrix} = (m-1)(n-1)\,I,$$

ou encore

$$\begin{vmatrix} y^2 & xy & x^2 \\ \phi''_x & \phi''_{xy} & \phi''_y \\ \psi''_x & \psi''_{xy} & \psi''_y \end{vmatrix} = (m-1)(n-1)\,I.$$

Or, élevons au carré, et multiplions par -2, il viendra

$$\begin{vmatrix} y^2 & -xy & x^2 \\ \phi''_x & \phi''_{xy} & \phi''_y \\ \psi''_x & \psi''_{xy} & \psi''_y \end{vmatrix} \begin{vmatrix} x^2 & 2\,xy & y^2 \\ \phi''_y & -2\,\phi''_{xy} & \phi''_x \\ \psi''_x & -2\,\psi''_{xy} & \psi''_y \end{vmatrix} = -2\,(m-1)^2(n-1)^2 I^2.$$

Posons maintenant

$$\left.\begin{aligned} \Phi &= x^2\phi''_x + 2\,xy\;\phi''_{xy} + y^2\phi''_y = m\,(m-1)\,\phi \\ \Psi &= x^2\psi''_x + 2\,xy\;\psi''_{xy} + y^2\psi''_y = n\;(n-1)\;\psi \\ \mathrm{K} &= \phi''_x\;\phi''_y - (\phi''_{xy})^2 \\ \mathrm{K}' &= \psi''_x\;\psi''_y - (\psi''_{xy})^2 \\ \mathrm{K}'' &= \phi''_x\;\psi''_y - 2\,\phi''_{xy}\;\psi''_{xy} + \phi''_y\;\psi''_x \end{aligned}\right\}. \tag{2}$$

On aura, en multipliant ligne par ligne,

$$2\,(m-1)^2(n-1)^2 I^2 = -\begin{vmatrix} 0 & \Phi & \Psi \\ \Phi & 2\,\mathrm{K} & \mathrm{K}'' \\ \Psi & \mathrm{K}'' & 2\,\mathrm{K}' \end{vmatrix}$$

d'où

$$(m-1)^2(n-1)^2 I^2 = -\,(\Phi^2\mathrm{K}' - \Phi\Psi\mathrm{K}'' + \Psi^2\mathrm{K}).$$

Mais si l'on appelle J, H, H', H'' le Jacobien, les Hessiens et l'Hessien simultané débarrassés de leur coefficients numériques, on aura

$$I = m^2nJ,\; K = m^2\,(m-1)^2\,H,\; K' = n^2\,(n-1)^2\,H',\; K'' = m^2n^2\,(m-1)^1(n-1)^2H'',$$

et par conséquent

$$J^2 = -\,(\phi^2H' - \phi\psi H'' + \psi^2H),^* \tag{3}$$

relation qu'il s'agissait de démontrer.

EXEMPLE. Soit $\phi = ax^2 + 2\,bxy + cy^2$, $\psi = a'x^2 + 2\,b'xy + c'y^2$, le coefficient de x^4 dans le J^2 de ces deux formes sera

$$(ab' - a'b)^2 = -\,\{a^2\,(a'c' - b'^2) - aa'\,(ac' + a'c - 2\,bb') + a'^2(ac - b^2)\},$$

comme il est aisé de vérifier.

4. *Pour faire Suite à la Démonstration du Jacobien.*

La formule que nous avons donnée suppose essentiellement que ϕ et ψ soient exprimés sous forme binomiale. Lorsque cela n'a pas lieu, et que les coefficients

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* M. Clebsch dans le second membre, au lieu du facteur -1, il trouve le facteur $-\frac{1}{2}$, probablement ce sera d'après quelques hypothèses faites sur les formes elles-mêmes. Mais notre formule laisse les formes et les Hessiens tels qu'on les entend généralement.

sont quelconques, alors il est aisé de voir en remontant à la démonstration que la formule deviendra celle-ci:

$$(m-1)^2(n-1)^2J^2 = -\{m^2(m-1)^2\phi^2H' - mn(m-1)(n-1)\phi\psi H'' + n^2(n-1)^2\psi H\}.$$

EXEMPLE 1. D'une forme cubique donnée, soient $C_{2,2}$, $C_{3,3}$ les covariants, Δ le discriminant. Alors on a

$$J = C_{3,3} \quad H' = -\Delta \quad \Delta = (ad-bc) - h(ac-b^2)(bd-c^2), H = 36\,C_{2,2}$$

et il viendra

$$C_{3,3}^2 = -(\Delta n^2 + hC_{2,2}^3).$$

relation connue (voir ma "Théorie des Formes Binaires").

EXEMPLE 2. Partons de la forme canonique de la quartique

$$u = x^4 + 6\gamma x^2y^2 + y^4.$$

Soit $v = \gamma x^4 + (1-3\gamma^2)x^2y^2 + \gamma y^4$ son covariant de 4-ordre, I_2, I_3 ses invariants quadratiques et cubiques. Nous poserons $\phi = u$, $\psi = v$, $J = 6\,hw$.

On trouve (voir même ouvrage p. 241) que l'Hessien de l'Hessien est

$$= 36\,I_3u - 12\,I_2v.$$

On trouve aussi que H'', l'Hessien simultané de u et de v est $= 1 + 3\gamma^2 = I_2u$. D'ailleurs H est l'Hessien de $u = v$. On aura donc pour le carré du covariant sextique de la quartique

$$81.6\,hw^2 = -[16.9.12\,u^2(3\,I_3u - I_2v) - 16.9.2\,hI_2u^2v + 16.9.12.12.v^3],$$

ou

$$w^2 = -[I_3u^3 - I_2u^2v. + hv^3],$$

ce qui est bien la relation cherchée entre les invariants et covariants fondamentaux de la quartique.

EXEMPLE 3. Soient $C_{2,2} = hx^2 + Mxy + Ny^2$, $C_{6,2} = Ex^2 + Fxy + by^2$ les covariants de 2^e^ ordre de la quintique, il viendra

$$\begin{vmatrix} 2\,hx + My, & Mx + 2\,Ny \\ 2\,Ex + Fy, & Fx + 2\,by \end{vmatrix}^2$$

$$= -h\{C_{2,2}^2(hEb - F^2) - C_{2,2}\,C_{3,3}(2\,hb + 2\,NE - MF) + C_{6,2}^2(4\,hN - M^2)\}.$$

Si l'on a recours aux formes canoniques de la quintique et de ses covariants, dues à M. Sylvester, il viendra en posant

$$\begin{aligned}
C_{2,2} &= acx^2 + (ac + bc - ab)\,xy + bcy^2,\\
C_{6,2} &= a^2b^2c^2\,(x^2 + xy + y^2),\\
I_4 &= a^2b^2 + b^2c^2 + c^2a^2 - 2\,abc\,(a + b + c),\\
I_3 &= a^2b^2c^2\,(ab + ac + bc),\\
I_{12} &= a^4b^4c^4,
\end{aligned}$$

cette relation algébrique

$$a^4b^4c^4\,\{(ab - bc)\,x^2 + 2\,xy\,(ac - bc)\,xy + (ac - ab)\,y^2\} = -\,\{3\,C_{2,2}^2\,I_{12} - 2\,C_{2,2}\,C_{3,3}\,I_8 - C_{6,2}^2\,I_4\}.$$

Si, au contraire, dans la formule générale ci-dessus, on prend pour ϕ et ψ les covariants $C_{2,2}\ C_{8,8}$, le Jacobien, à un coefficient numérique près, fournira $C_{5,3}$. On trouvera pareillement que le Hessien de $C_{2,2}$ est I_4; que le Hessien de $C_{3,3}$ est $4\,C_{6,2}$; que le Hessien simultané de $C_{2,2}$ et $C_{3,3}$ est $4\,C_{5,1}$, ou le covariant linéaire de 5ᵉ degré. Cela posé, on aura cette relation, que je crois nouvelle, entre quelques covariants de la quintique:

$$C_{5,3}^2 = 12\,C_{2,2}\,C_{8,8}\,C_{5,1} - 4\,C_{2,2}^2\,C_{6,2} - 9\,C_{3,3}^2\,I_4.$$

On conçoit qu'à l'aide de la formule, que nous avons démontré d'une façon directe, on pourra trouver beaucoup d'autres relations analogues.

5. *Sur la Résolution de la Quartique.*

On sait que toute quartique peut être réduite par une transformation linéaire à la forme canonique

$$x^4 + 6\,\mu x^2y^2 + y^4.$$

Or, en appellant δ le module de la transformation, on aura

$$I_2 = (1 + 3\,\mu^2)\,\delta^4,\quad I_3 = (\mu - \mu^3)\,\delta^6,\quad \Delta = I_2^3 - 27\,I_3^2 = (1 - 9\,\mu^2)^2\delta^{12}.$$

Il viendra par conséquent, en posant $\mu^2 = y$,

$$\frac{(1 + 3\,y)^3}{(1 - 9\,y)^2} = \frac{I_2^3}{\Delta} = k,$$

d'où
$$y^3 + (1 - 3\,k)\,y^2 + \frac{1 + 2\,k}{3}\,y + \frac{1 - k}{27} = 0.$$

De là on déduit que le discriminant Δ' de cette cubique est

$$\Delta' = \left(\frac{16}{27}\right)^2 k^2\,(1 - k)\,;$$

et comme $1 - k = -\,27\,\frac{I_3^2}{\Delta}$, il viendra $\Delta' = -\,\frac{16^2}{27}\,k^2\,\frac{I_3^2}{\Delta}$.

Or si $ax^3 + 3bx^2y + 3cxy^2 + dy^3$ représente une cubique, la racine est exprimée par

$$ax = -b + \sqrt[3]{\frac{A}{2} - \frac{a}{2}\sqrt{B}} + \sqrt[3]{\frac{A}{2} + \frac{a}{2}\sqrt{B}},$$

où $\quad B = (ab - bc)^2 - 4(ac - b^2)(bd - c^2),\ A = 3abc - a^2d - 2b^3,$

cette valeur de A donnant dans notre cas

$$A = \frac{18k(k-1)(3k-1) - 2k}{27}.$$

Ainsi on aura

$$\mu^2 = \frac{3k-1}{3} + \frac{1}{3}\sqrt[3]{k}\sqrt[3]{9(k-1)(3k-1) - 1 - 8\sqrt{1-k}}$$
$$+ \frac{1}{3}\sqrt[3]{k}\sqrt[3]{9(k-1)(3k-1) - 1 + 8\sqrt{1-k}};$$

d'où l'on déduit que μ^2 ne dépend que de l'invariant absolu $\frac{I_3^2}{\Delta}$.*

NOTE. Si $I_2^3 = \Delta,\ \mu = \sqrt{\frac{2}{3}}$

6. *Résolution de la Quintique dans le Cas ou $I_{18} = 0$.*

Soit l'équation canonique

$$lx^5 + my^5 - n(x+y)^5 = 0, \tag{1}$$

à laquelle correspond l'invariant

$$I_{18} = 4\,l^5m^5n^3(l-m)(l-n)(m-n).$$

Comme nous avons remarqué dans notre "Théorie des Formes Binaires," lorsque $I_{18} = 0$, les deux valeurs ou paramètres, auxquels peut se réduire la quintique, à savoir $\lambda = \frac{1}{n}, \mu = \frac{m}{n}$, sont égales; ainsi on peut poser l_{2m}, ce que par la valeur de I_{18} on pourrait prévoir. Alors la quintique (1) devient, en désignant par a une constante,

$$a(x^5 + 1) + (x+1)^5 = 0,$$

dont les racines sont évidemment réciproques. En enlevant la racine -1, il vient

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* Et que la résolution de la quartique peut être ramenée à celle d'une quadratique double à un paramètre dependant de l'invariant absolu $\frac{I_3^2}{\Delta}$.

$$(1+a)\,x^4 + x^3\,(4-a) + x^2\,(6+a) + x\,(4+a) + 1 + a = 0,$$

qui, en posant $x + \frac{1}{x} = z$, se transforme en

$$(1+a)\,z^2 + (h-a)\,z + 4 - 2 = 0,$$

d'où on tirera

$$x = \frac{a - 4 + \sqrt{5a\,(a-4} \pm \sqrt{-10a\,(6+a) + 2\,(a-4)\,\sqrt{5a\,(a-4)}}}{4\,(1+a)}. \qquad (2)$$

Si $a = 5$, on trouve pour les 5 racines

$$x_1 = -1, \qquad x_{2,3} = \frac{-1 \pm \sqrt{-35}}{6}, \qquad x_{4,5} = \frac{1 \pm \sqrt{-15}}{4}.$$

Et en général pour toute valeur de a, l'expression (2) fournira les 4 racines de la quintique en outre de la racine -1.

On General Differentiation.

By J. Hammond,
Buckhurst Hill, Essex, England.

1. Assuming that

$$D^n x^m = \frac{f(m+1)}{f(m-n+1)}\, x^{m-n}, \tag{1}$$

we see at once (on writing $n = 1$) that

$$f(m+1) = mf(m). \tag{2}$$

This result is given by nearly every writer on the subject, most of whom assume some particular value for $f(m)$. Of these assumptions that of Peacock is the simplest, viz. $f(m) = \Gamma(m)$, but Kelland shows that (2) is satisfied by other functions besides Gamma-Functions (Trans. Roy. Soc. Edinb., Vol. XIV. p. 578).

In fact, the general solution of (2) considered as a difference-equation is

$$f(m) = C_m \Gamma(m), \tag{3}$$

where $C_{m+1} = C_m$, and it may be noticed that Liouville's formula,

$$D^n x^m = (-)^n \frac{\Gamma(-m+n)}{\Gamma(-m)}\, x^{m-n},$$

corresponds to the particular solution

$$f(m) = (-)^{m-1} \frac{\sin m\pi}{\pi}\, \Gamma(m) = (-)^{m-1} \frac{1}{\Gamma(1-m)},$$

and that (3) may also be written in the form

$$f(m) = (-)^{m-1} \frac{C'_m}{\Gamma(1-m)}, \tag{4}$$

where $C'_{m+1} = C'_m$.

Now, A being any finite quantity which does not contain x, we have from (1),

$$D^n \Sigma A \frac{f(m-n+1)}{f(m+1)}\, x^m = \Sigma A x^{m-n},$$

and in the cases in which $\dfrac{f(m+1)}{f(m-n+1)} = \infty$,

$$D^n \,.\, 0 = \Sigma A x^{m-n}, \tag{5}$$

where the only conditions are that A is finite and independent of x, and that the summation extends to all values of m for which $\frac{f(m+1)}{f(m-n+1)}$ is infinite.

When n is a positive integer, the value of this fraction is, by (2),

$$m\,(m-1)\,\ldots\,(m-n+1),$$

a finite quantity; so that in this case (5) contains no terms, or there is no complementary function.

But when n is a negative integer $= -p$, the fraction is, by (3),

$$\frac{f(m+1)}{f(m+p+1)} = \frac{C_{m+1}\Gamma(m+1)}{C_{m+p+1}\Gamma(m+p+1)} = \frac{\Gamma(m+1)}{\Gamma(m+p+1)}$$

and is infinite for the values $-1, -2, -3, \ldots -p$ of m, and for no others; so that in this case (5) becomes

$$D^{-p} \,.\, 0 = A_1 x^{p-1} + A_2 x^{p-2} + \ldots + A_p.$$

Hence in every system of general differentiation we have, when n is a positive integer,

$$D^n \,.\, 0 = 0, \text{ and } D^{-n} \,.\, 0 = (1, x)^{n-1}.$$

The value of the complementary function may be obtained in a more definite form in each particular case; for example, if $f(m) = \Gamma(m)$ (Peacock's value), we have

$$\frac{A_1 x^{-1}}{\Gamma(0)} + \frac{A_2 x^{-2}}{\Gamma(-1)} + \frac{A\, x_{-3}}{\Gamma(-2)} + \ldots = 0;$$

and, operating with D^n,

$$D^n \,.\, 0 = \frac{A_1 x^{-1-n}}{\Gamma(-n)} + \frac{A_2 x^{-2-n}}{\Gamma(-1-n)} + \frac{A_3 x^{-3-n}}{\Gamma(-2-n)} + \ldots, \tag{6}$$

we obtain a form of the complementary function which satisfies the required conditions. This result is the same as (5), for $\frac{\Gamma(m+1)}{\Gamma(m-n+1)}$ can only become infinite for negative integral values of m.

Using Liouville's value of $f(m)$, the fraction $\frac{f(m+1)}{f(m-n+1)}$ becomes $(-)^n \frac{\Gamma(-m+n)}{\Gamma(-m)}$, which is only infinite for positive integral values of $m-n$, so that (5) contains in this case positive integral powers of x only.

Here again, since $f(m) = 0$ for positive integral values of m,

$$A_0 f(1)\, x^n + A_1 f(2)\, x^{n+1} + A_2 f(3)\, x^{n+2} + \ldots = 0;$$

operating, as before, with D^n, we get

$$D^n \,.\, 0 = A_0 f(n+1) + A_1 f(n+2)\, x + A_2 f(n+3)\, x^2 + \ldots, \tag{7}$$

a form of the complementary function such that, when n is a positive integer,

$$D^n \,.\, 0 = 0, \text{ and } D^{-n} \,.\, 0 = (1_1 x)^{n-1}.$$

As an additional example consider the case $f(m) = \cos 2\, m\pi\Gamma(m)$.

Here $\dfrac{\cos 2\, m\pi\Gamma(m+1)}{\cos 2\,(m-n)\,\pi\Gamma(m-n+1)}$ is infinite (1) when m is a negative integer, and (2) when $(m-n)$ is a fraction with an odd numerator, and with denominator 4; so that the complementary function is, by (5),

$$A_1 x^{-1-n} + A_2 x^{-2-n} + \ldots \ldots B_1 x^{\frac{1}{4}} + B_2 x^{\frac{3}{4}} + \ldots \ldots + C_1 x^{-\frac{1}{4}} + C_2 x^{-\frac{3}{4}} + \ldots \ldots$$

The first part of this must be of the form (6), and it is easy to show that the second and third parts each contain a factor $\sin n\pi$.

2. Whatever system of general differentiation we use, we see from (1) and (5) that whenever a term of the differentiated expression is similar to a term of the complementary function, its coefficient is infinite, and a correction must be used which will introduce logarithmic terms; in fact, when $\dfrac{f(m+1)}{f(m-n+1)}$ is infinite and h small,

$$D^n x^{m+h} = \frac{f(m+h+1)}{f(m+h-n+1)}\,(x^{m+h-n} - x^{m-n}),$$

the latter term being part of the complementary function,

$$= \frac{f(m+h+1)\, x^{m-n}}{f(m+h-n+1)} \,.\, h \log x \qquad \text{(in the limit).}$$

Or,
$$D^n x^m = x^{m-n} \log x \cdot \left[\frac{f(m+h+1)\, h}{f(m+h-n+1)}\right]_{h=0}.$$

In the first, or Peacock's case, putting $m = -\mu$ and $f = \Gamma$, this becomes, after some easy reductions,

$$D^n x^{-\mu} = \frac{(-)^{\mu-1} x^{-\mu-n} \log x}{(\mu-1)\,!\;\Gamma(-\mu-n+1)}. \tag{8}$$

Or, what is the same thing,

$$D^n x^{-\mu} = -\frac{\sin n\pi\Gamma(\mu+n)\, x^{-\mu-n} \log x}{\pi\,(\mu-1)\,!}. \tag{9}$$

In the second, or Liouville's case, putting $m - n = \mu$, we have

$$D^n x^{n+\mu} = \frac{f(n+\mu+1)}{\mu\,!\, f(0)}\, x^\mu \log x, \tag{10}$$

where in all three formulæ μ is and n is not a positive integer, and the complementary function is omitted; and in (10) $f(m)$ has either of the equivalent forms $(-)^{m-1}\dfrac{\sin m\pi\Gamma(m)}{\pi}$ or $(-)^{m-1}\dfrac{1}{\Gamma(1-m)}$.

If now, as a verification, n is put $= -\mu$ in (8) and $= -\mu - 1$ in (10), the results are

$$D^{-\mu} x^{-\mu} = \frac{(-)^{\mu-1}}{(\mu-1)!} \log x, \qquad \text{and } D^{-\mu-1} x^{-1} = \frac{x^{\mu}}{\mu!} \log x.$$

From the particular case $\mu = 0$ in (10) it is easy to deduce the values of $D^n \log x$ and $D^{-n} \log x$, which are found to be the same as those given by Greatheed (Camb. Math. Journ., Vol. I.) and Kelland (Trans. Roy. Soc. Edinb., Vol. XIV. p. 588), viz.

$$\left.\begin{array}{l} D^n \log x = (-)^{n-1}\,\Gamma(n)\, x^{-n} \\ D^{-n} \log x = (-)^n \dfrac{\pi}{\sin n\pi \Gamma(1+n)}\, x^n \end{array}\right\}. \tag{11}$$

3. But a more general value of $D^n \log x$ may be found as follows:

Since
$$D^n \frac{x^h - 1}{h} = \frac{f(h+1)\, x^{h-n}}{hf(h-n+1)} - \frac{f(1)\, x^{-n}}{hf(1-n)} + D^n . 0,$$

we have, taking the limit when $h = 0$,

$$D^n \log x = x^{-n} \left[\frac{f(h+1)\, x^h}{hf(h-n+1)} - \frac{f(1)}{hf(1-n)}\right]_{h=0} + D^n . 0.$$

Whence

$$D^n \log x = \frac{f(1)}{f(1-n)}\, x^{-n} \log x + \lambda_n x^{-n} + D^n . 0, \tag{12}$$

λ_n being written for $\left[\dfrac{f(h+1)}{hf(h-n+1)} - \dfrac{f(1)}{hf(1-n)}\right]_{h=0}$.

The value of λ_n may be found in various ways; in Liouville's case, $f(1) = 0$, and the value of λ_n is by (11) $(-)^{n-1}\Gamma(n)$, unless n be a negative integer; in Peacock's case, we have

$$D^n \log x = \frac{x^{-n} \log x}{\Gamma(1-n)} + \lambda_n x^{-n} + D^n . 0. \tag{13}$$

and

$$\lambda_n = \left[\frac{\Gamma(h+1)}{h\Gamma(h-n+1)} - \frac{\Gamma(1)}{h\Gamma(1-n)}\right]_{h=0} = \frac{1}{\Gamma(-n)} \int_0^1 \left[(1-x)^{-n-1} \frac{x^h - 1}{h}\right]_{h=0} dx$$
$$= \frac{1}{\Gamma(-n)} \int_0^1 \frac{\log x\, dx}{(1-x)^{n+1}},$$

and this determination is unsuitable when n is a positive integer.

The value of λ_n is easily seen to be $\dfrac{f'(1) f(1-n) - f(1) f'(1-n)}{\{f(1-n)\}^2}$, or it may be found from a difference-equation, as follows:

Writing (12) in the form

$$x^n D^n \log x = \frac{f(1)}{f(1-n)} \log x + \lambda_n + x^n D^n . 0,$$

and remembering that

$$(xD - n)\, x^n D^n = x^{n+1} D^{n+1},$$

we have

$$x^{n+1} D^{n+1} \log x = (xD - n) \left\{ \frac{f(1)}{f(1-n)} \log x + \lambda_n \right\} + x^{n+1} D^{n+1}.\, 0.$$

Therefore

$$\frac{f(1)}{f(-n)} \log x + \lambda_{n+1} = (xD - n) \left\{ \frac{f(1)}{f(1-n)} \log x + \lambda_n \right\};$$

or, since $f(1-n) = -nf(-n)$,

$$\lambda_{n+1} + n\lambda_n = \frac{f(1)}{f(1-n)}. \tag{14}$$

When n is any integer the functions f reduce to Gamma-Functions, so that, if n be positive and integral,

$$\left.\begin{array}{l} \lambda_{n+1} \;\; + n\lambda_n \;\; = 0 \\ \lambda_{-n+1} - n\lambda_{-n} = \dfrac{1}{n!} \end{array}\right\} \tag{15}$$

are the equations which respectively determine the coefficient of x^{-n} in the n^{th} differential and that of x^n in the n^{th} integral of $\log x$.

4. We cannot in general assume that $D^n e^{ax} = a^n e^{ax}$, but the complementary function can always be determined, so that

$$D^{n+1} e^{ax} = aD^n e^{ax}, \tag{16}$$

and with this value of the complementary function

$$D^n e^{ax} = C_n a^n e^{ax},$$

where C_n is any function of n subject to the conditions $C_{n+1} = C_n$ and $C_0 = 1$. For example, in Peacock's case,

$$D^n e^{ax} = \frac{x^{-n}}{\Gamma(1-n)} + \frac{ax^{1-n}}{\Gamma(2-n)} + \frac{a^2 x^{2-n}}{\Gamma(3-n)} + \ldots + D^n.\, 0, \tag{17}$$

the proper determination is

$$D^n.\, 0 = \frac{a^{-1} x^{-1-n}}{\Gamma(-n)} + \frac{a^{-2} x^{-2-n}}{\Gamma(-1-n)} + \frac{a^{-3} x^{-3-n}}{\Gamma(-2-n)} + \ldots;$$

and substituting this value of $D^n.\, 0$ in (17), we get an expression for $D^n e^{ax}$ which satisfies (16) and reduces to $a^n e^{ax}$ when n is either a positive or a negative integer.

From (17), omitting the complementary function and changing the sign of n, we deduce

$$D^{-n} e^x = \frac{x^n}{\Gamma(n+1)} + \frac{x^{n+1}}{\Gamma(n+2)} + \frac{x^{n+2}}{\Gamma(n+3)} + \ldots, \tag{18}$$

a series that may be summed as follows:

Let S denote the sum of the series, then

$$\frac{dS}{dx} = \frac{x^{n-1}}{\Gamma(n)} + S.$$

Solving this equation in the usual manner, we obtain

$$S = \frac{e^x}{\Gamma(n)} \int_0^x e^{-x} x^{n-1} dx,$$

the lower limit being determined from the circumstance that, n being positive, S vanishes with x.

Hence, restoring the complementary function, we have, when n is positive,

$$D^{-n} e^x = \frac{e^x}{\Gamma(n)} \int_0^x e^{-x} x^{n-1} dx + D^{-n}.0. \tag{19}$$

5. Expanding $(x+y)^m$ on the supposition $y < x$, and omitting the complementary function, we have

$$D_x^n (x+y)^m = D_x^n \left(x^m + m x^{m-1} y + \frac{m(m-1)}{1.2} x^{m-2} y^2 + \dots \right)$$

$$= \frac{f(m+1)}{f(m-n+1)} x^{m-n} + \frac{mf(m)}{f(m-n)} x^{m-n-1} y + \frac{m(m-1)f(m-1)}{1.2 f(m-n-1)} x^{m-n-2} y^2 + \dots$$

$$= \frac{f(m+1)}{f(m-n+1)} \left\{ x^{m-n} + (m-n) x^{m-n-1} y + \frac{(m-n)(m-n-1)}{1.2} x^{m-n-2} y^2 + \dots \right\}$$

$$= \frac{f(m+1)}{f(m-n+1)} (x+y)^{m-n} \tag{20}$$

$$D_y^n (x+y)^m = D_y^n \left(x^m + m x^{m-1} y + \frac{m(m-1)}{1.2} x^{m-2} y^2 + \dots \right)$$

$$= \frac{f(1)}{f(1-n)} x^m y^{-n} + \frac{mf(2)}{f(2-n)} x^{m-1} y^{1-n} + \frac{m(m-1)f(3)}{1.2.f(3-n)} x^{m-2} y^{2-n} + \dots$$

$$= f(1) \left\{ \frac{x^m y^{-n}}{f(1-n)} + \frac{m x^{m-1} y^{1-n}}{f(2-n)} + \frac{m(m-1) x^{m-2} y^{2-n}}{f(3-n)} + \dots \right\} \tag{21}$$

When n is a positive integer, (21) coincides with (20), which may be considered as the principal form of $D^n (x+y)^m$; when n is a negative integer, (21) becomes

$$\frac{x^m y^n}{n!} + m x^{m-1} \frac{y^{n+1}}{(n+1)!} + m(m-1) x^{m-2} \frac{y^{n+2}}{(n+2)!} + \dots$$

which is a form of the n^{th} integral of $(x+y)^m$ with respect to y, and again the two forms will be equivalent if the complementary function be properly chosen.

When n is fractional, (21) vanishes in Liouville's case, since $f(1) = 0$, but then (20), which contains only positive integral powers of y, forms part of $D_y^n . 0$.

But in Peacock's case $D_y^n . 0$ contains only negative fractional powers of y, so that no determination of the complementary function can make the two forms coincide. Hence we see that in general, when n is fractional, we cannot assume that

$$D_n^x \phi(x+y) = D_y^n \phi(x+y).$$

The two forms of $D^n (1+x)^m$ obtained from (20) and (21) are,

$x > 1$

$$D^n (1+x)^m = \frac{f(m+1)}{f(m-n+1)} (1+x)^{m-n} + D^n . 0, \tag{22}$$

$x < 1$

$$D^n (1+x)^m = f(1) \left\{ \frac{x^{-n}}{f(1-n)} + \frac{mx^{1-n}}{f(2-m)} + \frac{m(m-1)x^{2-n}}{f(3-n)} + \ldots . \right\} + D^n . 0; \tag{23}$$

and to these may be added two other formulæ obtained from (22) by considering the cases of failure as in (8) and (10), viz.

$x > 1$

$$D^n (1+x)^{-\mu} = \frac{(-)^{\mu-1} (1+x)^{-\mu-n} \log(1+x)}{(\mu-1)!\, \Gamma(-\mu-n+1)} + D^n . 0, \tag{24}$$

$$D^n (1+x)^{n+\mu} = \frac{f(n+\mu+1)}{\mu!\, f(0)} (1+x)^{\mu} \log(1+x) + D^n . 0, \tag{25}$$

the values of the letters being the same as in (8) and (10) respectively.

6. In Peacock's case, using formula (23) and writing $m = n = \frac{1}{2}$, we have

$x < 1$

$$\begin{aligned} D^{\frac{1}{2}} (1+x)^{\frac{1}{2}} &= \frac{x^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} + \frac{\frac{1}{2} x^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} + \frac{\frac{1}{2}(-\frac{1}{2}) x^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2}) x^{\frac{5}{2}}}{\Gamma(\frac{7}{2})} + \ldots . \\ &= \pi^{-\frac{1}{2}} \left(x^{-\frac{1}{2}} + x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{3} + \frac{x^{\frac{5}{2}}}{5} - \frac{x^{\frac{7}{2}}}{7} + \ldots . \right) \\ &= \pi^{-\frac{1}{2}} (x^{-\frac{1}{2}} + \tan^{-1} x^{\frac{1}{2}}); \end{aligned} \tag{26}$$

and if to verify we semi-differentiate again

$$\begin{aligned} D(1+x)^{\frac{1}{2}} &= \frac{x^{-1}}{\Gamma(0)} + \frac{\frac{1}{2} x^0}{\Gamma(1)} + \frac{\frac{1}{2}(-\frac{1}{2}) x}{\Gamma(2)} + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2}) x^2}{\Gamma(3)} + \ldots . \\ &= \tfrac{1}{2} \left\{ 1 - \tfrac{1}{2} x + (-\tfrac{1}{2})(-\tfrac{3}{2}) \frac{x^2}{2!} + \ldots . \right\} = \tfrac{1}{2} (1+x)^{-\frac{1}{2}}, \end{aligned}$$

no complementary function is here needed.

But using (22) in both Peacock's and Liouville's cases, the complementary function is required to correct our results; in fact, using the expanded form of (22), viz.

$x > 1$

$$D^n(1+x)^m = f(m+1)\left\{\frac{x^{m-n}}{f(m-n+1)} + \frac{x^{m-n-1}}{f(m-n)} + \frac{x^{m-n-2}}{2!f(m-n-1)} + \ldots\right\},$$

and putting $m = n = \frac{1}{2}$, this is

$$D^{\frac{1}{2}}(1+x)^{\frac{1}{2}} = f(\tfrac{3}{2})\left\{\frac{x^0}{f(1)} + \frac{x^{-1}}{f(0)} + \frac{x^{-2}}{2!f(-1)} + \frac{x^{-3}}{3!f(-2)} + \ldots\right\}$$

$= \Gamma(\frac{3}{2}) + 0$ in Peacock's case, and a complementary function is required, on semi-differentiating again, to supply the missing terms. In Liouville's case, on the other hand, none of the terms disappear, but the first term is infinite since $f(1) = 0$ and must be replaced by a logarithm by borrowing from the complementary function, as in Art. 2, so that in this case we have, after reducing,

$$D^{\frac{1}{2}}(1+x)^{\frac{1}{2}} = \frac{f(\frac{3}{2})}{f(0)}\left\{\log\psi + x^{-1} - \frac{x^{-2}}{2} + \frac{x^{-3}}{3} - \frac{x^{-4}}{4} + \ldots\right\}$$

$$= \frac{f(\frac{3}{2})}{f(0)}\log(1+x), \tag{27}$$

which agrees with formula (25).

For a further verification of (27) it may be noticed that when $m = \frac{1}{2}$ and $n = \frac{3}{2}$ in (22), omitting the complementary function we have

$$D^{\frac{3}{2}}(1+x)^{\frac{1}{2}} = \frac{f(\frac{3}{2})}{f(0)}(1+x)^{-1},$$

which may also be deduced by differentiating (27).

7. From (26) it is easy to obtain

$$D^{\frac{1}{2}}\tan^{-1}x^{\frac{1}{2}} = \Gamma(\tfrac{3}{2})(1+x)^{-\frac{1}{2}}, \tag{28}$$

Peacock's formula being used and the complementary function omitted; and this is a particular case of a more general theorem, for if

$$\phi(x) = A_1x + \frac{A_3x^3}{3!} + \frac{A_5x^5}{5!} + \frac{A_7x^7}{7!} + \ldots$$

$$D^{\frac{1}{2}}\phi(x^{\frac{1}{2}}) = D^{\frac{1}{2}}\left(A_1x^{\frac{1}{2}} + \frac{A_3x^{\frac{3}{2}}}{3!} + \frac{A_5x^{\frac{5}{2}}}{5!} + \ldots\right)$$

$$= \Gamma(\tfrac{3}{2})\left\{A_1 + \frac{A_3 3x}{2.3!\Gamma(2)} + \frac{A_5 5.3x^2}{2^2.5!\Gamma(3)} + \frac{A_7 7.5.3x^3}{2^3.7!\Gamma(4)} + \ldots\right\}$$

$$= \Gamma(\tfrac{3}{2})\left(A_1 + \frac{A_3x}{2^2} + \frac{A_5x^2}{2^2.4^2} + \frac{A_7x^3}{2^2.4^2.6^2} + \ldots\right), \tag{29}$$

and in particular, if $A_1 = 1$, $A_3 = 1^2$, $A_5 = 1^2.3^2$, $A_7 = 1^2.3^2.5^2$, ... (29) becomes

$$D^{\frac{1}{2}}\sin^{-1}x^{\frac{1}{2}} = \Gamma(\tfrac{3}{2})\left(1 + \frac{1^2}{2^2}x + \frac{1^2.3^2}{2^2.4^2}x^2 + \frac{1^2.3^2.5^2}{2^2.4^2.6^2}x^3 + \ldots\right)$$

$$= \frac{\Gamma(\frac{1}{2})}{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - x \sin^2\theta}} = \pi^{-\frac{1}{2}} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - x \sin^2\theta}},$$

and a similar expression can be found for $\int_0^{\frac{\pi}{2}} \sqrt{1 - x \sin^2\theta}\, d\theta$, viz. in this case $\phi(x) = \int_0^x (1 - x^2)^{\frac{1}{2}}\, dx$, while in the former case $\phi(x) = \int_0^x \frac{dx}{\sqrt{1 - x^2}}$, and throughout $x < 1$.

8. A proof of Leibnitz's Theorem, which does not assume $D^n e^{ax} = a^n e^{ax}$, will conclude the present paper.

Let u, v be two functions of x capable of expansion in powers of x, and suppose that

$$u = \Sigma A_m x^m, \qquad v = \Sigma B_n x^n; \qquad \text{then } uv = \sum A_m B_n x^{m+n}$$

[this last summation extending to all values of m which occur in u, and of n in v].

Hence
$$D^\mu uv = D^\mu . 0 + \sum A_m B_n D^\mu x^{m+n}$$
$$= D^\mu . 0 + \sum A_m B_n \frac{f(m+n+1)}{f(m+n-\mu+1)} x^{m+n-\mu}. \tag{30}$$

Now, omitting all complementary functions, it will be shown that the series

$$D^\mu u . v + \mu D^{\mu-1} u . v' + \frac{\mu(\mu-1)}{1.2} D^{\mu-2} u . v'' + \dots .$$

is equal to the second term of (30); for the series is

$$\sum A_m B_n \left\{ \frac{f(m+1)}{f(m-\mu+1)} x^{m-\mu} x^n + \mu \frac{f(m+1)}{f(m-\mu+2)} x^{m-\mu+1} n x^{n-1} \right.$$
$$\left. + \frac{\mu(\mu-1)}{1.2} \cdot \frac{f(m+1)}{f(m-\mu+3)} x^{m-\mu+2} n(n-1) x^{n-2} + \dots . \right\},$$

which, by (3),

$$= \sum A_m B_n \frac{C_m}{C_{m-\mu}} \frac{x^{m+n-\mu}}{\Gamma(-\mu)} \left\{ \frac{\Gamma(m+1)\Gamma(-\mu)}{\Gamma(m-\mu+1)} - n \frac{\Gamma(m+1)\Gamma(1-\mu)}{\Gamma(m-\mu+1)} \right.$$
$$\left. + \frac{n(n-1)}{1.2} \frac{\Gamma(m+1)\Gamma(2-\mu)}{\Gamma(m-\mu+3)} - \frac{n(n-1)(n-2)}{1.2.3} \frac{\Gamma(m+1)\Gamma(3-\mu)}{\Gamma(m-\mu+4)} + \dots . \right\}$$

$$= \sum A_m B_n \frac{C_m}{C_{m-\mu}} \frac{x^{m+n-\mu}}{\Gamma(-\mu)} \int_0^1 (1-x)^m x^{-\mu-1} \left(1 - nx + \frac{n.n-1}{1.2} x^2 \dots . \right) dx$$

$$= \sum A_m B_n \frac{C_m}{C_{m-\mu}} \frac{x^{m+n-\mu}}{\Gamma(-\mu)} \cdot \frac{\Gamma(m+n+1)\Gamma(-\mu)}{\Gamma(m+n-\mu+1)};$$

and if v only contains integral powers of x, n is integral and $C_m = C_{m+n+1}$; so that this expression reduces to the same form as (30).

Hence, if u be capable of expansion in any series of powers of x, and v in a series of integral powers,

$$D^{\mu}uv = D^{\mu}.0 + \left(D^{\mu}u.v + \mu D^{\mu-1}u.v' + \frac{\mu(\mu-1)}{1.2}D^{\mu-2}u.v'' + \ldots\right).$$

The restriction of the nature of the expansion of v to *positive integral* powers of x is attended with this advantage that the coefficient of $x^{m+n-\mu}$ in the series for $D^{\mu}uv$ always consists of a finite number of terms, viz. $n+1$.

Note on a Theorem for Expanding Functions of Functions.

BY EMORY MCCLINTOCK, F.I.A., *Milwaukee, Wisconsin.*

IT has just come to my knowledge that the "theorem for expanding functions of functions" lately published by me (Vol. II. p. 348) was essentially anticipated, twenty years earlier, by Mr. S. Roberts, F.R.S. Mr. Roberts's brief, yet sufficient statement (Quarterly Journal, Vol. IV. p. 195) is as follows, putting $\phi x = a_0 + a_1 x + a_2 x^2 + \ldots$, and $\Pi = a_1 \frac{d}{da_0} + 2a_2 \frac{d}{da_1} + 3a_3 \frac{d}{da_2} + \ldots :$—

"It will be observed that, since $\Pi^n \phi x = D_x^n \phi x$, the notation of (1) applies generally to a function of ϕx and the differentials of ϕx, and we may write

$$F(\phi, \phi', \phi'', \ldots) = \epsilon^{x\Pi} F(a_0, a_1, 2a_2, \ldots)."$$

MILWAUKEE, Aug. 3, 1880.

Notes on Relative Motion.

By James Loudon, *University College, Toronto.*

1. Motion of a point in a plane.

At time t let the moving axes be $O\xi$, $O\eta$, and P a point (ξ, η) in their plane. At time $t + \delta t$ let these axes coincide with $O\xi'$, $O\eta'$, and P with P'; then the ξ and η components of the displacement PP' are $-\omega\eta\delta t$, $\omega\xi\delta t$, respectively, if ω is the rate at which the axes turn round $O\zeta$. Let a moving point be at P at time t, and at Q at time $t + \delta t$, the co-ordinates of Q referred to $O\xi'$, $O\eta'$ being $\xi + \dot{\xi}\delta t$, $\eta + \dot{\eta}\delta t$; then the absolute velocity of the moving point is ultimately $\frac{PQ}{\delta t} = \left(\frac{PP'}{\delta t}, \frac{P'Q}{\delta t}\right)$, the ξ and η components of which are $\dot{\xi} - \omega\eta$, $\dot{\eta} + \omega\xi$, respectively.

Putting $\dot{\xi} - \omega\eta = u = OA$, and $\dot{\eta} + \omega\xi = v = OB$, the component velocities at time $t + \delta t$ become $u + \dot{u}\delta t = OA'$ along $O\xi'$, and $v + \dot{v}\delta t = OB'$ along $O\eta'$. Hence the absolute acceleration ultimately $= \left(\frac{AA'}{\delta t}, \frac{BB'}{\delta t}\right)$, the components of which are

$$\dot{u} - v\omega = \ddot{\xi} - 2\,\omega\dot{\eta} - \eta\dot{\omega} - \omega^2\xi \text{ along } O\xi,$$

$$\dot{v} + u\omega = \ddot{\eta} + 2\,\omega\dot{\xi} + \xi\dot{\omega} - \omega^2\eta \text{ along } O\eta.$$

2. Motion of a rigid body round a fixed axis $O\zeta$, the axes $O\xi$, $O\eta$ being fixed in the body.

At time t the whole momentum is $-M\omega\eta = OA$ along $O\xi$, and $M\omega\xi = OB$ along $O\eta$, where ξ, η are co-ordinates of the centre of inertia. At time $t + \delta t$ the momentum is $-M\eta\,(\omega + \dot{\omega}\delta t) = OA'$ along $O\xi'$, and $M\xi\,(\omega + \dot{\omega}\delta t) = OB'$ along $O\eta'$. The changes of momentum per unit time are, therefore, ultimately $\frac{AA'}{\delta t}, \frac{BB'}{\delta t}$, whose components are

$$-M\eta\dot{\omega} - M\omega^2\xi \text{ along } O\xi,$$

$$M\xi\dot{\omega} - M\omega^2\eta \text{ along } O\eta.$$

At time t the whole moment of momentum is (employing OA, OB in a new sense)

$$-\beta\omega = OA \text{ along } O\xi,$$
$$-\alpha\omega = OB \text{ along } O\eta,$$
$$C\omega \;.\;.\;.\;. \text{ along } O\zeta,$$

where $$\alpha = \Sigma m\eta\zeta, \quad C = \Sigma m(\xi^2 + \eta^2), \text{ etc.}$$

At time $t + \delta t$ the moment of momentum becomes

$$-\beta(\omega + \dot{\omega}\delta t) = OA' \text{ along } O\xi',$$
$$-\alpha(\omega + \dot{\omega}\delta t) = OB' \text{ along } O\eta', \text{ etc.}$$

Hence the changes per unit time of moment of momentum are ultimately $\frac{AA'}{\delta t}$, $\frac{BB'}{\delta t}$, $C\dot{\omega}$, the components of which are $-\beta\dot{\omega} + \alpha\omega^2$ along $O\xi$, $-\alpha\dot{\omega} - \beta\omega^2$ along $O\eta$, and $C\dot{\omega}$ along $O\zeta$.

These, it will be observed, are of the same form as when the axes are fixed in space.

3. To measure the absolute velocity and acceleration of a point referred to axes moving in space round O.

Let the motion of the axes be due to rotations θ_1, θ_2, θ_3 measured along themselves. Then, proceeding as in § 1, the displacements of a point $P(\xi, \eta, \zeta)$ due to these rotations are $(\zeta\theta_2 - \eta\theta_3)\,\delta t$ along $O\xi$, $(\xi\theta_3 - \zeta\theta_1)\,\delta t$ along $O\eta$, and $(\eta\theta_1 - \xi\theta_2)\,\delta t$ along $O\zeta$. These added to the relative displacements $(\dot{\xi}\delta t, \dot{\eta}\delta t, \dot{\zeta}\delta t)$ of the moving point give the absolute displacements. Hence the components of the absolute velocity are

$$u = OA = \dot{\xi} + \zeta\theta_2 - \eta\theta_3 \text{ along } O\xi,$$
$$v = OB = \dot{\eta} + \xi\theta_3 - \zeta\theta_1 \text{ along } O\eta,$$
$$w = OC = \dot{\zeta} + \eta\theta_1 - \xi\theta_2 \text{ along } O\zeta.$$

Again, let the velocities at time $t + \delta t$ be $OA' = u + \dot{u}\delta t$ along $O\xi'$, etc.; then the absolute accelerations are ultimately $\frac{AA'}{\delta t}$, $\frac{BB'}{\delta t}$, $\frac{CC'}{\delta t}$, whose components are

$$\dot{u} - v\theta_3 + w\theta_2 \text{ along } O\xi,$$
$$\dot{v} - w\theta_1 + u\theta_3 \text{ along } O\eta,$$
$$\dot{w} - u\theta_2 + v\theta_1 \text{ along } O\zeta.$$

These become, on reduction,

$$\ddot{\xi} - 2\,\theta_3\dot{\eta} + 2\,\theta_2\dot{\zeta} + \zeta\dot{\theta}_2 - \eta\dot{\theta}_3 - (\theta_1^2 + \theta_2^2 + \theta_3^2)\,\xi + (\xi\theta_1 + \eta\theta_2 + \zeta\theta_3)\,\theta_1$$

along $O\xi$, etc.

NOTE. — These resolutions are most readily effected as follows: AA' is equivalent to AD along $O\eta$, DH along $O\zeta$, and HA' along $O\xi'$; and similar resolutions are effected for BB', CC'. The values of AD, DH, etc. are at once derived from the displacements in time δt of the points (1, 0, 0), (0, 1, 0), (0, 0, 1). The latter are, respectively,

$$\begin{array}{ccc} 0\,, & \theta_3\,, & -\theta_2\,, \\ -\theta_3\,, & 0\,, & \theta_1\,, \\ \theta_2\,, & -\theta_1\,, & 0\,, \end{array}$$

each multiplied by δt; from which the values of AD, DH, etc. are obtained by multiplying the first set by OA, the second by OB, and the third by OC. Moreover, the parts HA', etc. remain unchanged in magnitude when resolved along $O\xi$, $O\eta$, $O\zeta$, if infinitesimals above the first order be neglected. Thus, in the present case, $HA' = \dot{u}\delta t$, $AD = u\theta_3\delta t$, $DH = -u\theta_2\delta t$.

4. If, in the previous case, the origin moves, its acceleration must of course be added to the expressions found in § 3. These formulas may be tested by the following well-known example. Let O be on the earth's surface in latitude λ, and let $O\xi$ be drawn south, $O\eta$ east, and $O\zeta$ vertical. Then ω being the earth's rotation and r its radius, the accelerations of O are

$$\begin{array}{ll} -\,\omega^2 r \cos\lambda \sin\lambda \text{ along } O\xi, \\ -\,\omega^2 r \cos^2\lambda \quad \text{“} \quad O\zeta. \end{array}$$

Also, $\theta_1 = -\,\omega \cos\lambda$, $\theta_2 = 0$, $\theta_3 = \omega \sin\lambda$, and $\dot{\theta}_1 = 0 = \dot{\theta}_2 = \dot{\theta}_3$.

Hence the accelerations of m at (ξ, η, ζ) are

$$\ddot{\xi} - \omega^2 r \cos\lambda \sin\lambda - 2\,\omega\dot{\eta} \sin\lambda - \omega^2\xi \sin^2\lambda - \omega^2\zeta \sin\lambda \cos\lambda,$$
$$\ddot{\eta} + 2\,\omega\dot{\zeta} \cos\lambda + 2\,\omega\dot{\xi} \sin\lambda - \omega^2\eta,$$
$$\ddot{\zeta} - \omega^2 r \cos^2\lambda - 2\,\omega\dot{\eta} \cos\lambda - \omega^2\zeta \cos^2\lambda - \omega^2\xi \sin\lambda \cos\lambda,$$

along $O\xi$, $O\eta$, $O\zeta$, respectively.

5. To measure the changes in the rotation of a rigid body with one point fixed, the axes moving as in § 3. Let the rotations to which the displacement of the body is due be at time t, $\omega_1 = OA$, $\omega_2 = OB$, $\omega_3 = OC$ measured respectively along $O\xi$, $O\eta$, $O\zeta$. Then since at time $t + \delta t$ these become $\omega_1 + \dot{\omega}_1\delta t = OA'$, etc. along $O\xi'$, $O\eta'$, $O\zeta'$, the absolute changes per unit time in the rotation are ultimately

$$\frac{AA'}{\delta t}, \frac{BB'}{\delta t}, \frac{CC'}{\delta t}.$$

Resolving these, we get for the required components

$$\dot{\omega}_1 - \omega_2\theta_3 + \omega_3\theta_2 \text{ along } O\xi, \text{ etc.}$$

6. To measure the change in the whole absolute momentum of a rigid body, one point of which is fixed at O, the axes moving as in §§ 3, 5.

Since the absolute momentum of m in the position (ξ, η, ζ) at time t is

$$m\,\{\zeta(\omega_2+\theta_2)-\eta(\omega_3+\theta_3)\} \text{ along } O\xi, \text{ etc.},$$

it follows that the whole absolute momentum at that time is

$$\begin{aligned} &z(\omega_2+\theta_2)-y(\omega_3+\theta_3) \text{ along } O\xi,\\ &x(\omega_3+\theta_3)-z(\omega_1+\theta_1) \text{ along } O\eta,\\ &y(\omega_1+\theta_1)-x(\omega_2+\theta_2) \text{ along } O\zeta,\end{aligned}$$

each multiplied by M, where (x, y, z) is the position of the centre of inertia. Calling these components $\mu_1 = OA$, $\mu_2 = OB$, $\mu_3 = OC$, respectively, it follows that at time $t+\delta t$ they become $\mu_1+\dot\mu_1\,\delta t = OA'$ along $O\xi'$, $\mu_2+\dot\mu_2\,\delta t = OB'$ along $O\eta'$, $\mu_3+\dot\mu_3\delta t = OC'$ along $O\zeta'$. The changes in the whole momentum per unit time are, therefore, $\frac{AA'}{\delta t}, \frac{BB'}{\delta t}, \frac{CC'}{\delta t}$, whose components are

$$\begin{aligned} &\dot\mu_1-\mu_2\,\theta_3+\mu_3\,\theta_2 \text{ along } O\xi,\\ &\dot\mu_2-\mu_3\,\theta_1+\mu_1\,\theta_3 \text{ along } O\eta,\\ &\dot\mu_3-\mu_1\,\theta_2+\mu_2\,\theta_1 \text{ along } O\zeta.\end{aligned}$$

Since $\dot x = z\omega_2 - y\omega_3$, etc. these expressions become, on reduction, M times

$$\begin{aligned} z(\dot\omega_2+\dot\theta_2)-y(\dot\omega_3+\dot\theta_3)+\omega_1\,\{(\omega_1+\theta_1)\,x+(\omega_2+\theta_2)\,y+(\omega_3+\theta_3)\,z\}\\ +(\omega_1+\theta_1)\,(\theta_1 x+\theta_2 y+\theta_3 z)-x\,\{(\omega_1+\theta_1)^2+(\omega_2+\theta_2)^2+(\omega_3+\theta_3)^2\}\end{aligned}$$

for the first, with similar values for the other two.

7. To measure the changes in the whole absolute moment of momentum under the same circumstances as in § 6. Since the absolute moment of m's momentum at time t is m times

$$(\omega_1+\theta_1)\,(\eta^2+\zeta^2)-(\omega_2+\theta_2)\,\xi\eta-(\omega_3+\theta_3)\,\zeta\xi \text{ along } O\xi,$$

with corresponding components along $O\eta$, $O\zeta$, it follows that the components of the whole moment of momentum at that time are

$$\begin{aligned} &A(\omega_1+\theta_1)-\gamma(\omega_2+\theta_2)-\beta(\omega_3+\theta_3) \text{ along } O\xi,\\ -&\gamma(\omega_1+\theta_1)+B(\omega_2+\theta_2)-\alpha(\omega_3+\theta_3) \text{ along } O\eta,\\ -&\beta(\omega_1+\theta_1)-\alpha(\omega_2+\theta_2)+C(\omega_3+\theta_3) \text{ along } O\zeta,\end{aligned}$$

where $\quad A = \Sigma m\,(\eta^2+\zeta^2), \quad \alpha = \Sigma m\eta\zeta,$ etc.

Let these components be called $\nu_1 = OA$, $\nu_2 = OB$, $\nu_3 = OC$, respectively. Then at time $t + \delta t$ they become $\nu_1 + \dot\nu_1\delta t = OA'$ along $O\xi'$, $\nu_2 + \dot\nu_2\delta t = OB'$ along $O\eta'$, and $\nu_3 + \dot\nu_3\delta t = OC'$ along $O\zeta'$. Hence the changes of the moment of momentum per unit time are

$$\frac{AA'}{\delta t}, \frac{BB'}{\delta t}, \frac{CC'}{\delta t},$$

whose components are

$$\begin{aligned}
&\dot\nu_1 - \nu_2\theta_3 + \nu_3\theta_2 \text{ along } O\xi,\\
&\dot\nu_2 - \nu_3\theta_1 + \nu_1\theta_3 \text{ along } O\eta,\\
&\dot\nu_3 - \nu_1\theta_2 + \nu_2\theta_1 \text{ along } O\zeta.
\end{aligned}$$

Now, since $\dot\xi = \zeta\omega_2 - \eta\omega_3$, etc., it follows that

$$\begin{aligned}
\dot A &= 2\,\Sigma m\,(\eta\dot\eta + \zeta\dot\zeta)\\
&= 2\,(\gamma\omega_3 - \beta\omega_2)\\
\dot B &= 2\,(a\omega_1 - \gamma\omega_3)\\
\dot C &= 2\,(\beta\omega_2 - a\omega_1)\\
\dot a &= \Sigma m\,(\eta\dot\zeta + \zeta\dot\eta)\\
&= (C - B)\,\omega_1 - \gamma\omega_2 + \beta\omega_3\\
\dot\beta &= \gamma\omega_1 + (A - C)\,\omega_2 - a\omega_3\\
\dot\gamma &= -\,\beta\omega_1 + a\omega_2 + (B - A)\omega_3.
\end{aligned}$$

Hence the above values for the component changes of moment of momentum become

$$A\,(\dot\omega_1 + \dot\theta_1) - \gamma\,(\dot\omega_2 + \dot\theta_2) - \beta\,(\dot\omega_3 + \dot\theta_3) + 2\,(\omega_1 + \theta_1)\,(\gamma\omega_3 - \beta\omega_2) - (\omega_2 + \theta_2)\,[-\,\beta\omega_1 + a\omega_2 + (B - A)\,\omega_3] - (\omega_3 + \theta_3)\,[\gamma\omega_1 + (A - C)\,\omega_2 - a\omega_3] - \theta_3\,[-\,\gamma\,(\omega_1 + \theta_1) + B\,(\omega_2 + \theta_2) - a\,(\omega_3 + \theta_3)] + \theta_2\,[-\,\beta\,(\omega_1 + \theta_1) - a\,(\omega_2 + \theta_2) + C\,(\omega_3 + \theta_3)]$$

for the first; with similar expressions for the other two.

On Certain Ternary Cubic-Form Equations.

BY J. J. SYLVESTER.

CHAPTER I.

EXCURSUS C. — ON THE TRISECTION AND QUARTISECTION OF THE ROOTS OF UNITY TO A PRIME-NUMBER INDEX.

WHAT follows, so far as it relates to the trisection of the primitive roots of unity, may be regarded as auxiliary to Postscriptum 2, Vol. II. of this Journal, p. 387, inasmuch as it establishes the equation in ω which, when $x = \frac{\omega - 1}{3}$, becomes the equation assumed in line 4, p. 388. The rest is episodical, except so far as it may be regarded as correlative to the subject matter of Titles 1 and 2 of Excursus A.*

It will be seen that the equations to a system of three and four periods, usually obtained by long and tedious processes, may, with the aid of one simple and well-known principle, be deduced by processes almost elementary in their character, and into which enter no algebraical calculations except of the very easiest kind.

A sketch of the method was laid by me before the Scientific Congress held at Rheims in the month of August last.

The index p of the roots is, as usual, supposed to be a prime number; e is the number of the periods, f the number of roots whose sum forms a period, so that $ef = p - 1$; the periods themselves will be called η, viz. $\eta_1, \eta_2, \ldots \eta_e$.

Preliminaries.

1. I say, in the first place, that the sum of the i^{th} powers of the periods will be congruous to $-f^{i-1}$ in respect to the modulus p.

* In any future redistribution of the contents of the entire memoir, it would be proper to incorporate the matter contained in Proscriptum 2, pp. 387–389, with this Excursus.

For, were it not that in the development of the i^{th} power of any one of the η's some of the combinations of the powers of the roots were unity, it is obvious that we should have $\Sigma\,\eta^i = -\,ef^i \div (p-1)$, i. e. $-f^{i-1}$, and that we might regard every term in such development as equivalent to $-\frac{1}{p-1}$, without affecting this result. The existence of terms equal to unity will render it necessary to substitute for any such term 1 instead of $-\frac{1}{p-1}$, in order to obtain a correct result, and if there be N of them, the correction to be introduced will be $N\left(1+\frac{1}{p-1}\right)$, i. e. $\frac{N}{p-1}\cdot p$; but as it is obvious that the result must be an integer, it follows that N must be double by $(p-1)$, and consequently the value of $\Sigma\,\eta^i$ *to modulus* p will be $-f^{i-1}$, i. e. $-\left(\frac{p-1}{e}\right)^{i-1}$, as was to be shown.

2. From the above it follows that *to modulus* p,

$$\Sigma\,(e\eta+1)^i \equiv (-1)^i + e(-1)^{i-1} + e\frac{e-1}{2}(-1)^{i-2} +, \text{etc.}, \equiv (-1+1)^e \equiv 0,$$

or, in other words, $\Sigma\,(e\eta+1)^i$ is divisible by p.

But, if s_i and σ_i represent, respectively, the sum of the i^{ary} combinations and i^{ary} powers of the roots of an equation, we know that $(-)^i s_i =$ coefficient of x^i in $e^{-\sigma_1 x - \frac{\sigma_2}{2}x^2 - \frac{\sigma_3}{3}x^3 \cdots}$, so that s_i multiplied by numbers none exceeding i, is expressible as the sum of integer multiples of $\sigma_\lambda \sigma_\mu \sigma_\nu \ldots.$ where $\lambda+\mu+\nu+\ldots. = i$.

3. Consequently, s_i multiplied by integers none greater than i, when the roots in question are the e values of $e\eta+1$ and $i>0$, will be divisible by p, and consequently, since e is less than p, all the coefficients of the equation to which those roots appertain will be divisible by p, the first, of course (which is unity), excepted.

Since $\Sigma\,(e\eta+1) = e\Sigma\eta + e = 0$, the equation whose roots are $\omega_1, \omega_2, \ldots. \omega_e$ where $\omega = e\eta+1$ will be of the form $\omega^e + P\omega^{e-2} + Q\omega^{e-3} +$, etc., where P, Q, etc., each contain p; and I may remark, incidentally (although the fact is immaterial to the object in view), that, as may easily be seen, $\Sigma\omega^i$ will be divisible not only by p but also by e, and that consequently the coefficient of ω^{e-i}, in the above equation, will contain the greatest common divisor to e and i.

4. The coefficient P has one or the other of two determinate algebraical values according as f, i. e. $\frac{p-}{e}$, is even or odd.

In the former case, the congruence $x^e + 1 \equiv 0$ [mod p] is soluble, and in the latter, insoluble. Accordingly, in the latter case, we shall have $\Sigma\eta^2 = -f$, and in the former $\Sigma\eta^2 = p - f$, and in each case $\Sigma\eta^2$ will be an odd number. Also,

when f is odd (which involves the necessity of e being even) $\Sigma\omega^2 = \Sigma\,(e\eta + 1)^2 = -e^2\frac{p-1}{e} - 2e + e = -ep$, and when f is even $\Sigma\omega^2$ will be this result augmented by e^2p, i. e. $(e^2 - e)p$.

Consequently, $P = \frac{e}{2}p$, or $= -\frac{e^2-e}{2}p$, according as f is odd or even.

Thus, when $e = 3$, f being necessarily even, $P = -3p$, and when $e = 4$, $P = -6p$, or $= 2p$, according as $\frac{p-1}{4}$ is even or odd.*

5. With regard to what immediately follows it will also be necessary to determine the *form* of Q in respect to certain moduli for the cases of e equal to 3 and e equal to 4. In the former case $\Sigma\omega^3 = \Sigma\,(e\eta + 1)^3 = \Sigma\,(e^3\eta^3 + 3\,e^2\eta^2 + 3\,e\eta + 1) \equiv 3 \bmod 9$, and consequently, since $Q = -\frac{1}{3}\,\Sigma\omega^3$, $-3\,Q \equiv 3 \bmod 9$ and $-Q \equiv 1 \bmod 3$.

In the latter case, i. e. when $e = 4$, since $\Sigma\eta^2$ is always odd $\Sigma\omega^3$ [to mod 32] $\equiv 16 - 12 + 4$, i. e. $\equiv 8$, and, consequently, $-3\,Q \equiv 8$ to that modulus.

These *preliminaries* being established, I will now proceed to state the principle referred to in the exordium.

Principle.

A rational integer function of any set of periods of the roots of unity whose coefficients are all whole numbers, which does not change its value for a circular substitution executed upon the periods, it is well known, must be an integer number; but to this I add that if such function, without changing its arithmetical value, undergoes a change of sign when such a substitution is made, it must necessarily be an integer number multiplied by the difference of the two periods into which the entire sum of the roots may be divided, that is to say, will be a multiple of $\sqrt{p}$, when p is of the form $4\,K + 1$ and of $\sqrt{-p}$, when p is of the form $4\,K - 1$.†

As an example, the product of the differences of the roots of the equation in η will be an integer number when e, the number of the periods, is odd and an integer number multiple of $\sqrt{p}$ or $\sqrt{-p}$ (according as $\frac{p-1}{2}$ is even or odd), when the number of periods is even. As another example, if $e = 2\,\epsilon$, the function

$$(\eta_0 - \eta_\epsilon)\,(\eta_1 - \eta_{\epsilon+1})\,(\eta_2 - \eta_{\epsilon+2}) \ldots . (\eta_{\epsilon-1} - \eta_{2\epsilon-1})$$

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* When $e = 2$, $P = p$ or $-p$ according as f is odd or even, so that the equation in ω takes the known form $\omega^2 \pm p = 0$.

† To put the matter more clearly, call the alternating function F and the difference spoken of Δ. Then ΔF is invariable in sign as well as in magnitude for the circular substitutions in question. Hence $F = \frac{\text{An Integer}}{\sqrt{\pm p}}$ but F^2 is an Integer; therefore $F =$ An Integer $\sqrt{\pm p}$. Q. E. D.

which changes its sign but not its quantitative value, when $0, 1, 2, 3, \ldots . (2\epsilon - 1)$ are replaced by $1, 2, 3, \ldots . -1, 0$ will be an integer multiple of $\sqrt{p}$, or of $\sqrt{-p}$, according as ϵ is even or odd.

We are now in a position to obtain without difficulty the well-known equivalent to the equation corresponding to $e = 3$, given at p. 388, and the corresponding pair of equations for the case of $e = 4$.

A. *Case of* $e = 3$.

The equation in ω, from what has been shown in the preliminaries, must be of the form $\omega^3 - 3px + pq = 0$, and it only remains to determine q.

The discriminant of the above equation being $q^2p^2 - 4p^3$, it follows that the product of the differences of its roots will be $27(4p^3 - q^2p^2)$. But this product is 3^6 into $(\eta_0 - \eta_1)^2 (\eta_0 - \eta_2)^2 (\eta_1 - \eta_2)^2$, which latter, by the *principle*, is of the form M^2. We have, therefore,

$$4p^3 - q^2p^2 = 27M^2 = 27m^2p^2.$$

Hence, $$4p = q^2 + 27m^2,$$

which serves to determine the value of q^2 absolutely.

To find the value of q, it follows from the preliminaries that $qp \equiv -1 \bmod 3$, and, consequently, since $p \equiv 1 \bmod 3$, $q \equiv -1 \bmod 3$, so that q is perfectly determined.

B. *Case of* $e = 4$.

$\omega^2 - 2\sqrt{p}\omega + R = 0$, $\omega^2 + 2\sqrt{p}\omega + R' = 0$, will be the form of the equations containing, respectively, the pairs of roots ω_0, ω_2 and ω_1, ω_3; for

$$\omega_0 + \omega_2 = (4\eta_0 + 1) + (4\eta_2 + 1) = 2(2\overline{\eta_0 + \eta_2} + 1) = 2(2\delta_0 + 1),$$

and, similarly, $\omega_1 + \omega_3 = 2(2\overline{\eta_1 + \eta_3} + 1) = 2(2\delta_1 + 1)$ where δ_0 and δ_1 are the two periods which make up together the sum of all the roots, so that $2\delta_0 + 1$ and $2\delta_1 + 1$ are the roots of the equation $\Omega^2 - p = 0$, the sign of the last term being fixed from the fact of $\frac{p-1}{2}$ being by hypothesis even.

Furthermore, R, R' must be of the form $Ap + B\sqrt{p}, Ap - B\sqrt{p}$; for $(R - R')\sqrt{p}$, being integer, requires that R, R' shall be of the form $A_1 + B\sqrt{p}, A_1 - B\sqrt{p}$, and then RR' being an integer multiple of p involves the necessity of A_1^2, and therefore of A_1 containing p.

The product $(\eta_0 - \eta_2)(\eta_1 - \eta_3)$ consequently becomes

$$((A-1)p + B\sqrt{p})\ ((A-1)p - B\sqrt{p}),$$

which by the principle must be of the form m^2p, and consequently,

$$(A-1)^2p-B^2=C^2 \text{ or } (A-1)^2p=B^2+C^2.$$

The coefficient of ω^2 becomes $-4p+2Ap$ which, by the preliminaries, when $\frac{p-1}{4}$ is even must be equal to $-6p$, so that $A=-1$, and when $\frac{p-1}{4}$ is odd must be equal to $2p$, so that $A=3$.

In each case, therefore, $(A-1)^2=4$ and $4p=B^2+C^2$; consequently, if $p=g^2+h^2$, $4g^2=B^2$, and $4h^2=C$, and the complete equation in ω containing the roots $\omega_0, \omega_1, \omega_2, \omega_3$, becomes $(\omega^2-p)^2-4p(\omega+g)^2=0$ when $\frac{p-1}{4}$ is even and $(\omega^2+3p)^2-4p(\omega+g)^2=0$ when $\frac{p-1}{4}$ is odd. In either case g^2 is given, but the sign of g requires to be determined; alike, however, for one case as for the other, $-8pg$ being the 3d coefficient after the first, we must have, as shown in the preliminaries, $24pg \equiv 8$ [mod 32], and consequently, since p is of the form $4K+1$, $24g \equiv 8$ [mod 32]. Hence, $3g \equiv 1$ [mod 4], i. e. $g \equiv -1$ [mod 4], which gives the required complete determination of g.

The quartisecting equations thus naturally arrived at are expressed in the form in which, according to Bachmann (Kreistheilung, p. 230), they were first presented by Lebesgue; the method here given for finding the equations for the trisection and quatrisection of the roots of unity will be found on examination to be incomparably simpler, shorter, and more direct than any in common use, and as removing a serious stumbling-block from the path of the student, and, occurring, so far as regards trisection, in the natural course of the development of my subject, I have thought entitled to a place in this memoir. Why I require the trisecting equation is, as will be remembered, to enable me to obtain the conditions of 2 and of 3 being cubic residues to a given index. The condition for 2 being such, strange to say, is nowhere to be found in Bachmann's Kreistheilung, although the cubic character of 3 is there duly and fully made out.

The conditions of the one and of the other being cubic residues were, I am informed by M. Lucas, given for the first time in a letter from Gauss to Mlle. Sophie Germain.

EXCURSUS B.

TITLE 5 (*bis*). — *On the Law of Squares.*

There being errors and inaccuracies not a few in the matter printed under this title, owing to my absence abroad as it went through the press, I have thought it desirable to rewrite it, rectifying the errors, and supplying some steps which were wanting in the demonstrations.* I shall, in what follows, use through-

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* In the postscript which was thought out on board the transatlantic steamer, the Bothnia, and written out, as far as I can recollect, at a single sitting a day or two before posting it at Queenstown, I have not been able to detect any inaccuracy in the results, although some additional steps and explanations might advantageously have been supplied.

There is, perhaps, one slight exception to be made to this statement as regards the very important theorem, stated but not proved, concerning the nature of the form $X\xi + Y\eta + Z\zeta$ where the coefficients of ξ, η, ζ are supposed to be the reduced co-ordinates of any derivative to x, y, z. If $U = 0$ is the equation to the cubic in its general form, obviously X, Y, Z are indeterminate, as each may be augmented by an arbitrary multiple of U of suitable degree and order. Consequently, the theorem ought to have been stated in the following form. The co-ordinates X, Y, Z of any such derivative *may be* so expressed that $X\xi + Y\eta + Z\zeta$ shall be a mixed concomitant to U. The fundamental invariantive concomitants to a ternary cubic involving not more than one system of cogredients and a single linear system of contragredients are eleven in number and of the types underwritten:

4 . 0 . 0	4 . 4 . 1
6 . 0 . 0	5 . 4 . 1
1 . 3 . 0	7 . 4 . 1
3 . 3 . 0	9 . 7 . 1
8 . 6 . 0	11 . 7 . 1
12 . 9 . 0	

Hence the co-ordinates of every rational derivative in the natural scale to a point on a cubic curve may be expressed as the coefficients of the contragredient variables in a rational integer function of the above eleven quantities linear in the latter five, and such that its degree and orders for the n^{th} grade are $\frac{4(n^2-1)}{3}$; n^2, 1.

The particular forms of X, Y, Z which appertain to the concomitant $X\xi + Y\eta + Z\zeta$, and which may be called the *normal* forms, it may be added, are those which actually arise from the processes of *colligation* and *reduction* described in the excursus. By *colligation* I mean the determination of the *general* analytical connective of x, y, z; x', y', z' by the same method as that applied at pages 61, 62, to the canonical quadrinomial form of the cubic. The co-ordinates of such connective are absolutely determinate, inasmuch as the equation which each set of co-ordinates must satisfy is of the order 3, whereas the co-ordinates in question are of the second order only in each set of variables (and of course of the first degree in the coefficients of the cubic). By *reduction* I mean that when in the co-ordinates of the general connective for x, y, z; x', y', z' are substituted the *normal* forms of the co-ordinates for derivatives of the grades μ, $\mu + 3i$, their common factor of the degorder $(12i^2 - 3, 9i^2)$ is to be cast out.

This common factor, it may be noticed, is *always* a covariant of the cubic. When $i = 1$, it is seen *a posteriori* that this is the case, for its value is expressible (see footnote, p. 87) under the form of a known covariant, say Θ (which was obtained by means of using the canonical form of the cubic); that it must be true for all values of i may be deduced from the general algebraical theorem that if in a covariant to any given form, in place of the variables x, y, z be substituted $\frac{d\Omega}{d\xi}, \frac{d\Omega}{d\eta}, \frac{d\Omega}{d\zeta}$ where Ω is any invariantive concomitant to such form, and ξ, η, ζ are contragredient to x, y, z, the resulting expression will be itself an invariantive concomitant. To obtain now the reducing factor for the connective to P_μ, $P_{\mu+3i}$ (p. 88) it is only necessary to substitute in Θ x_i, y_i, z_i (the normal co-ordinates of the i^{th} derivative) in lieu of x, y, z where $x_i\xi + y_i\eta + z_i\zeta$ is known to be an invariantive concomitant to the cubic. Hence, by the algebraical theorem above stated, the corresponding reducing factor (not containing ξ, η, ζ) is necessarily a covariant to the cubic, as was to be shown.

out P_i to denote the i^{th} derivative of P, and x_i, y_i, z_i to signify the reduced co-ordinates of P_i, so that P_1, x_1, y_1, z_1 will mean the same as P, x, y, z respectively. $x+y=0$, $z=0$ will be taken as the auxiliary point of inflexion, serving to complete the scale, and will be called I. In the natural scale it is easy to see that any derived co-ordinate, as z_i, must contain the original one, as z. For when $z=0$, P will be a point of inflexion and P_i identical with P, hence (x_i, y_i, z_i) will express the same point of inflexion, and consequently $z_i = 0$; hence z_i must contain z. When we leave the rational scale, so that i is a multiple of 3, z must contain xyz. For when $z=0$, the i^{th} derivative P will be one of the three points I, I', I'', expressed by $z=0$, $x^3+y^3=0$. If P is I, P_3 is obviously I; if P is I', P_2 is I', and P_3 will be the connective of P_2 and I''; consequently P_3 is I and $z=0$, and the same will be the case if P is I''; hence z_3' contains z.

Again, if $y=0$, P will be some inflexion J, and the connective to I, J being called K, P_3 will be the connective of J, K, i. e. I, as before; hence z_3 will contain y, and in like manner it will contain x. Also, since in each case P_3 is I, every derivative of P_3 will be I; hence, when $xyz=0$, $z_{3\nu}$ becomes 0; consequently z_i (if i is a multiple of 3) contains xyz.

Again, if x_i, y_i, z_i are the reduced co-ordinates of P_i, I say that $x_i(y_i^3 - z_i^3)$; $y_i(z_i^3 - x_i^3)$; $z_i(x_i^3 - y_i^3)$ will be the *reduced* co-ordinates of x_{2i}, y_{2i}, z_{2i}.

For, if possible, let two of the above co-ordinates have a common factor F; then, since x_i, y_i, z_i have no common factor, $x_i^3 - y_i^3$, $y_i^3 - z_i^3$ have a common factor, and when $F=0$, $x_i^3 = y_i^3 = z_i^3$; but $x_i^3 + y_i^3 + z_i^3 + Kx_iy_iz_i = 0$. Hence, unless $x_i^3 = y_i^3 = z_i^3 = 0$, we must have $3 + \sqrt[3]{1}\,K = 0$, but K is arbitrary. Hence, F must be contained in x_i, y_i, z_i contrary to hypothesis.

Although it is a consequence of a general law* that z_i cannot contain z^2, for present purposes it will be sufficient to establish that z_i cannot, for each of two consecutive values of i, contain z^2. Thus, suppose z_{2i-1} and z_{2i} each contained z^2, then, because z_{2i} contains z^2, z_i must do so too; since, otherwise, $x_i^3 - y_i^3$ must contain z. If that is possible, let $z=0$; then $x_i^3 - y_i^3 = 0$; but P, and therefore P_i, becomes an inflexion, whereas $x_i^3 = y_i^3$ is the necessary and sufficient condition that P_i is a Plückerian point, which is self-contradictory. But since z_i contains z^2, z_{i-1} must also contain z^2, for z_{2i-1} will be contained (see p. 82) in $\frac{1}{z}(x_iy_iz_{i-1}^2 - x_{i-1}y_{i-1}z_i^2)$, and therefore, if z_{i-1} does not contain z^2, z must be contained in x_i or y_i, which is impossible. In like manner, if z^2 is contained in

* The law is that $x^i.y^i.z^i\,x\,y_iz_i$, cannot for any value of i contain a square algebraical factor, just as and *en dernière analyse* for the same general kind of reason that the binomial exponential $(a^i + b^i)$ can contain no such factor.

z_{2i}, z_{2i-1}, it will be contained also in z_i and z_{i-1}. Hence it would be contained eventually in z, which is absurd.

Again, it may be shown that z will be the only common measure to z_{i-1} and z_i. For, if possible, let them have any other common measure F, and let F become zero. Then P_{i-1} and P_i both become points of inflexion belonging to the system previously designated as I, I', I'', and by a collineation process performed on these points alone or combined with I, P may be obtained. Hence P belongs to the same system of inflexions, i. e. $z = 0$. Hence F would be contained in a power of z, contrary to hypothesis.

I will now show that if the two systems of unreduced co-ordinates obtained by the colligation of

$$\left.\begin{matrix} x_{i-1}, & y_{i-1}, & z_{i-1} \\ x_i, & y_i, & z_i \end{matrix}\right\} \text{ and of } \left\{\begin{matrix} y_{i-1}, & x_{i-1}, & z_{i-1} \\ x_i, & y_i, & z_i \end{matrix}\right.$$

be called F, G, H; F', G', H; respectively, the terms F, G, H; F', G', H can have no other measure common to all four than z, or, in other and more precise terms, z is the greatest common measure to the greatest common measures of F, G, H and of F', G', H. For brevity call the two sets of co-ordinates of P_{i-1} and P_i, u, v, w; u', v', w' respectively. Then the unreduced co-ordinates in question will be (p. 82)

$$\left.\begin{aligned} F &= vwu'^2 - v'w'u^2 \\ G &= wuv'^2 - w'u'v^2 \\ H &= uvw'^2 - u'v'w^2 \end{aligned}\right\} \text{ and } \left\{\begin{aligned} uwu'^2 - u'w'v^2 &= F' \\ wvv'^2 - w'v'u^2 &= G' \\ vuw'^2 - v'u'w^2 &= H \end{aligned}\right.$$

into each of which z necessarily enters as a factor, because w, w' have been proved each to contain z.

[u, v, it will be observed, cannot have a common factor, for then u, v, w would have a common factor contrary to hypothesis; and, in like manner, u', v' can have no common factor.]

I say, in the first place, that no indecomposable function of x, y, z, say M, not contained either in w or in w', can be common to F, G, F', G'. For, if so, let F vanish; then, calling $\frac{u}{w}, \frac{v}{w}$; $\frac{u'}{w'}, \frac{v'}{w'}$, r, s; r', s' respectively, we have

$$\begin{aligned} &(1)\quad sr'^2 - s'r^2 = 0, \qquad && rr'^2 - s's^2 = 0, \quad (3) \\ &(2)\quad rs'^2 - r's^2 = 0, \qquad && r'r^2 - ss'^2 = 0. \quad (4) \end{aligned}$$

Now, none of the terms r, s, r', s' can vanish: e. g. r cannot vanish, for, if so, from (1) it would follow that $s = 0$, or $r' = 0$, and from (3) that $s = 0$, or $s' = 0$, so that either $r = 0$ and $s = 0$, or $r' = 0$ and $s' = 0$, i. e. the general values of u and v or of u' and v' must have a common factor M, which is impossible. Hence,

combining (1) and (2) or (3) and (4), we derive $rs = r's'$ (5), as might also be obtained immediately by equating to zero the term common to the two systems above.

From (5), from (3) and (4), and from (1) and (2) we obtain respectively

$$r^3s^3 = r'^3s'^3, \quad r^3r'^3 = s^3s'^3, \quad r'^3s^3 = r^3s'^3,$$

the second and third of which are equivalent to $r^6 = s^6$, $r'^6 = s'^6$, and the first and second combined give $r'^6 = s^6$. Hence $r^6 = r'^6 = s^6 = s'^6$, and consequently the original equations (1), (2), (3) give $r^3 = s^3 = r'^3 = s'^3$.

The equations $r^3 = s^3$, $r'^3 = s'^3$ imply that P_{i-1}, P_i are each of them distinct or identical antitangentials to one of the points of inflexion corresponding to $z = 0$, i. e. are each of them a Plückerian point on the cubic, and P or (P, I) will be a residual either to P_{i-1}, P_i or to $(P_{i-1}, I), P_i$ where I is the auxiliary inflexion used to complete the scale. Hence P is either a Plückerian or an inflexion point, and in either case P_2 will necessarily be an inflexion. Hence one at least of the derivatives P_{i-1}, P_i is an inflexion, but each is a Plückerian, which is absurd.

Thus M (an irresoluble factor common to F, G, F', G') must be contained either in w or in w'. Suppose it is not z and is contained in w, then it cannot be contained in w', for w, w' have no common measure except z, and consequently when $M = 0$, $v'u^2 = 0$, $u'v^2 = 0$, $v'v^2 = 0$, and $u'u^2 = 0$, and either u and v or u' and v' each become zero, which is impossible seeing that neither the general values of u, v nor those of u', v' can have any common factor. In like manner, it follows that M cannot be contained in w'. Consequently, the two systems F, G, H; F', G', H can have no other common measure, except some power of z.

Finally, I say that the only common measure in question is z itself. 1°. Suppose it were possible (which it is not) that one of the two terms w or w' (say w) contains z^2, then it has been proved that the other (w') cannot contain z^2. Hence, if $wvv'^2 - w'u'u^2$ contains z^2, u or u' must contain z, and in like manner, if w' and not w contained z, v or v' must contain z, none of which suppositions are admissible.

2°. Suppose that neither w nor w' contains z^2. Then writing $w = \omega z$, $w' = \omega' z$, and writing for $\frac{u}{\omega}, \frac{v}{\omega}; \frac{u'}{\omega'}, \frac{v'}{\omega'}, r, s; r', s'$ respectively, we shall obtain over again, as before, $r^3 = s^3$, $r'^3 = s'^3$, indicating as before that P_{i-1} and P_i are each of them Plückerian points when $z = 0$, i. e. when P is a point of inflexion, which is doubly absurd. Hence it follows that the common measures of F, G, H and of F', G', H have the common measure z, and no other.

We are now in a position to prove the *law of squares.* Suppose it is true for P_{i-1} and P_i, I say it will be true for P_{2i-1}. For consider the connectives of

$$\left.\begin{matrix} x_{i-1}, & y_{i-1}, & z_{i-1} \\ x_i, & y_i, & z_i \end{matrix}\right\} \text{ and of } \left\{\begin{matrix} y_{i-1}, & x_{i-1}, & z_{i-1} \\ x_i, & y_i, & z_i \end{matrix}\right.$$

as expressed by the formulas above employed. Let $z^2\Omega$ be the third term common to the unreduced systems of co-ordinates.

Allowing (as is the fact) that Ω does not contain z, the reducing factor common to the unreduced co-ordinates of P (or it may be its opposite in respect to I) must be $z\Omega$, and consequently to the other system corresponding to P_{2i-1} or its opposite, can only be z or z^2; but the latter is impossible, for then z_{2i-1} would not contain z.

Again, if Ω could be conceived equal to $z^q\Omega_1$, the reducing factor for P or its opposite would be $z^{1+q}\Omega_1$, and consequently that for P_{2i-1} or its opposite could not be z^2 and would be z as before. Hence the order of P_{2i-1} in the variables is necessarily $2(i-1)^2+2i^2-1$, i. e. $4i^2-4i+1$ or $(2i-1)^2$.

Moreover, it has been shown that if x_i, y_i, z_i are the reduced co-ordinates for P_i, $x_i(y_i^3-z_i^3)$, $y_i(z_i^3-x_i^3)$, $z_i(x_i^3-y_i^3)$ are such for P_{2i}, and consequently, if the law is true for i, it is true for $2i$. Hence, being true for 1, it is true for 2, and therefore for 3, and therefore for 4 and 5 and 6, and therefore for $3+4$, i. e. 7, and for 2.4, i. e. 8, and for $4+5$, i. e. 9, and for 2.5, i. e. 10, and so on for every number, as was to be proved.* Thus, this negative proposition, as I have termed it (p. 85), is completely established. There remains to prove the important proposition contained (but incorrectly proved) on pages 84, 85, to wit, that the unreduced systems of co-ordinates arising from the colligation of

$$\left.\begin{matrix} (x_i, y_i, z_i) \\ (x_j, y_j, z_j) \end{matrix}\right\} \text{ and of } \left\{\begin{matrix} (y_i, x_i, z_i) \\ (x_j, y_j, z_j) \end{matrix}\right.$$

will be of the forms LN', MN', NN'; $L'N, M'N, N'N$, where L, M, N; L', M', N' are the reduced systems of the co-ordinates of the connectives of P_i, P_j, and P'_i, P_j respectively.

To illustrate this proposition by an example, consider the connectives of P', P_3, i. e. P_2 and of P, P_3, i. e. P_4.

z_2 is $z(y^3-x^3)$ and z_4 is of the form $z(y^3-x^3)\,\Omega$, where Ω is of the order 12 in the variables.

* In other words, if the theorem is true up to i inclusive, any number between $i+1$ and $2i$ inclusive is either of the form $2j$ or $2j-1$, where j does not exceed i; and being true for j, it is true for $2j$, and being true for $j-1$ and j, it is true for $2j-1$. Hence, if true up to i it is true up to $2i$, but it is true for $i=1$ and therefore for all values of i. Q. E. D.

Call X_4, Y_4, Z_4 the unreduced co-ordinates arising from the colligation of P, P_3. Suppose $x^3 - y^3$ to become zero, then P becomes a Plückerian, and P_3 will be also such, viz. one of the nine appertaining to the inflexions given by $z = 0$.* Hence $x_3^3 - y_3^3$ becomes zero. Now X_4, Y_4 represent $yzx_3^2 - y_3z_3x^2$, $xzy_3^2 - x_3z_3y^2$ respectively, and since $yzx_3^2 \,.\, x_3z_3y^2 - y_3z_3x^2 \,.\, xzy_3^2 = zz_3\,(x_3^3y^3 - y_3^3x_3^3)$ $= 0$, $X_4 : Y_4 :: yx_3^2 : xy_3^2$, and consequently $X_4^3 - Y_4^3 = 0$; but P_4 is a point of inflexion and not a Plückerian; hence X_4, Y_4 must each contain the factor $x^3 - y^3$, and Z_4 must be of the form $z^2\,(x^3 - y^3)^2\Omega$, for after division by $z\,(x^3 - y^3)$ it must still contain that factor. Also X_4, Y_4, Z_4 can have no other common measure except $z\,(x^3 - y^3)$, for after throwing out that factor the quotient is of the order 16, the order of z_4 given by the law of squares. Thus we see that the third unreduced coefficient common to (P, P_3) and (P', P_3) is equal to $z_2 \,.\, z_4$, as it ought to be according to the proposition in question.

To be Continued.

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* The nine points of inflexion on a cubic curve form a closed group, but so also do any three of them which lie in a right line, and also any single one. In like manner, the nine inflexions with their antitangentials, any three of these lying in a right line with their antitangentials, and any one with its antitangentials, form closed groups containing 36, 12, and 4 points respectively. The ornamental-gardening problem of *alignement*, anglicè *allineation*, which consists in so disposing a number of points on a plane as to obtain the maximum number or all the various possible numbers of right lines each containing three of the points, finds its systematic solution in the theory of groups of inflexional and sub-inflexional points of various grades. In some very old numbers of the "Educational Times" will be found questions of the kind proposed by me (not reproduced in the Reprint), of which the solution depends on this order of considerations. In certain cases that had been studied, I ascertained the possible existence of a larger number of collineations than had previously been imagined by other writers on the subject, among whom Mr. S. B. Woolhouse deserves special mention for the ingenuity of his constructions. As far as I am aware, the theory of allineation has never been treated by other writers than myself, except by empirical methods, and its dependence on the theory of the general cubic curve was not even suspected.

Change of the Independent Variable.

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BY Taylor's Theorem, if $u \equiv f(y)$ and $x \equiv \phi(y)$,

$$f(y+h) = u + hd_y u + h^2 \frac{d_y^2 u}{2!} + h^3 \frac{d_y^3 u}{3!} + \text{etc.}; \qquad \text{(A)}$$

also
$$f(y+h) = f\phi^{-1}[x + \{\phi(y+h) - x\}]$$

$$= u + \{\phi(y+h)-x\} d_x u + \{\phi(y+h)-x\}^2 \frac{d_x^2 u}{2!} + \{\phi(y+h)-x\}^3 \frac{d_x^3 u}{3!} + \text{etc.} \qquad \text{(B)}$$

$$\phi(y+h) = x + hd_y x + h^2 \frac{d_y^2 x}{2!} + h^3 \frac{d_y^3 x}{3!} + \text{etc.} \qquad \text{(C)}$$

Substitute by (C) for $\{\phi(y+h) - x\}$ in (B), and equate coefficients of like powers of h in (A) and (B) thus reduced.

For convenience, let

$$x_1 \equiv d_y x, \qquad x_2 \equiv \frac{d_y^2 x}{2!}, \qquad x_3 \equiv \frac{d_y^3 x}{3!}, \ldots .$$

$$u_1 \equiv d_y u, \qquad u_2 \equiv \frac{d_y^2 u}{2!}, \qquad u_3 \equiv \frac{d_y^3 u}{3!}, \ldots .$$

and $S_m^n \equiv$ the sum of the terms of weight m in the expansion of

$$(0_0 + x_1 + x_2 + x_3 + \ldots .)^n$$

(from which it immediately follows that, if $p > m$, $S_m^p = 0$ and that $S_m^m = x_1^m$).

$$u_1 = x_1 d_x u,$$

$$u_2 = x_2 d_x u + x_1^2 \frac{d_x^2 u}{2!},$$

$$u_3 = x_3 d_x u + 2\,x_1 x_2 \frac{d_x^2 u}{2!} + x_1^3 \frac{d_x^3 u}{3!},$$

$$u_4 = x_4 d_x u + (2\,x_1 x_3 + x_2^2) \frac{d_x^2 u}{2!} + 3\,x_1^2 x_2 \frac{d_x^3 u}{3!} + x_1^4 \frac{d_x^4 u}{4!},$$

$$u_5 = x_5 d_x u + 2\,(x_1 x_4 + x_2 x_3) \frac{d_x^2 u}{2!} + 3\,(x_1^2 x_3 + x_1 x_2^2) \frac{d_x^3 u}{3!} + 4\,x_1^3 x_2 \frac{d_x^4 u}{4!} + x_1^5 \frac{d_x^5 u}{5!},$$

and generally

$$u_n = S'_n . d_x u + S^2_n . \frac{d^2_x u}{2!} + S^3_n . \frac{d^3_x u}{3!} + S^4_n . \frac{d^4_x u}{4!} + \dots + S^n_n . \frac{d^n_x u}{n!};$$

$$\therefore\ d_x u = \frac{u_1}{x_1}; \qquad \frac{d^2_x u}{2!} = \frac{\begin{vmatrix} x_1 & u_1 \\ x_2 & u_2 \end{vmatrix}}{x_1 . x_1^2};$$

$$\frac{d^3_x u}{3!} = \frac{\begin{vmatrix} x_1 & 0 & u_1 \\ x_2 & x_1^2 & u_2 \\ x_3 & 2\,x_1 x_2 & u_3 \end{vmatrix}}{x_1 . x_1^2 . x_1^3}; \qquad \frac{d^4_x u}{4!} = \frac{\begin{vmatrix} x_1 & 0 & 0 & u_1 \\ x_2 & x_1^2 & 0 & u_2 \\ x_3 & 2\,x_1 x_2 & a_1^3 & u_3 \\ x_4 & 2\,x_1 x_3 + x_2^2 & 3\,x_1^2 x_2 & u_4 \end{vmatrix}}{x_1 . x_1^2 . x_1^3 . x_1^4};$$

$$\frac{d^5_x u}{5!} = \frac{\begin{vmatrix} x_1 & 0 & 0 & 0 & u_1 \\ x_2 & x_1^2 & 0 & 0 & u_2 \\ x_3 & 2\,x_1 x_2 & x_1^3 & 0 & u_3 \\ x_4 & 2\,x_1 x_3 + x_2^2 & 3\,x_1^2 x_2 & x_1^4 & u_4 \\ x_5 & 2\,(x_1 x_4 + x_2 x_3) & 3\,(x_1^2 x_3 + x_1 x_2^2) & 4\,x_1^3 x_2 & u_5 \end{vmatrix}}{x_1 . x_1^2 . x_1^3 . x_1^4 . x_1^5};$$

and generally

$$\frac{d^n_x u}{n!} = \frac{\begin{vmatrix} S^1_1 & & & & & u_1 \\ S^1_2 & S^2_2 & & & & u_2 \\ S^1_3 & S^2_3 & S^3_3 & & & u_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ S^1_{n-1} & S^2_{n-1} & S^3_{n-1} & & S^{n-1}_{n-1} & u_{n-1} \\ S^1_n & S^2_n & S^3_n & & S^{n-1}_n & u_n \end{vmatrix}}{x_1 . x_1^2 . x_1^3 \dots x_1^n},$$

which may be written

$$\frac{d^n_x u}{n!} = \{ S^1_1 . S^2_2 . S^3_3 \dots S^{n-1}_{n-1} . u_n - S^1_1 . S^2_2 . S^3_3 \dots S^{n-2}_{n-2} . u_{n-1} | S^{n-1}_n | + S^1_1 . S^2_2 . S^3_3 \dots S^{n-3}_{n-3} . u_{n-2} | S^{n-2}_{n-1}, S^{n-1}_n |$$
$$\dots (-)^{p-1} S^1_1 . S^2_2 . S^3_3 \dots S^{n-p}_{n-p} . u_{n-p+1} | S^{n-p+1}_{n-p+2}, S^{n-p+2}_{n-p+3}, \dots S^{n-1}_n | + \dots \} \div S^1_1 . S^2_2 . S^3_3 \dots S^n_n .$$

If $u = y$, this reduces to the last term, i. e., since in this case $u_1 = 1$, to

$$\frac{d^n_x u}{n!} = (-)^{n-1} \mid S^1_2, S^2_3, S^3_4, \dots S^{n-1}_n \mid \div S^1_1 . S^2_2 . S^3_3 \dots S^n_n .$$

Note on the Intersections of Two Curves.

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In Salmon's "Higher Plane Curves" (p. 16), the theorem is given that if of the n^2 points of intersection of two curves of the n^{th} order np lie on a curve of the p^{th} order, a curve of the $(n-p)^{\text{th}}$ order may be passed through the remaining $n(n-p)$ of the points. It can likewise be proved that if of the mn intersections of two curves of the m^{th} and n^{th} orders, np lie on a curve of the p^{th} order (where $m > n$ and $m > p$), a curve of the $(m-p)^{\text{th}}$ order can be passed through the remaining $n(m-p)$ intersections. For through any $\frac{1}{2}(m-p)(m-p+3)$ of the remaining points we can pass a curve of the $(m-p)^{\text{th}}$ order which, with the curve of the p^{th} order, forms a curve of the m^{th} order, of whose points of intersection with the curve of the n^{th} order

$$np + \tfrac{1}{2}(m-p)(m-p+3)$$

coincide with those of the given curves of the m^{th} and n^{th} orders. If, then, this number is equal to or greater than the number of points required to determine all the intersections of a curve of the m^{th} with a curve of the n^{th} order, the composite curve of the m^{th} order will pass through all the intersections of the given curves, and hence the curve of the $(m-p)^{\text{th}}$ order will pass through all but the given np of them which lie on the curve of the p^{th} order. The proposition is therefore true if

$$np + \tfrac{1}{2}(m-p)(m-p+3) \gtreqless mn - \tfrac{1}{2}(m-1)(m-2),$$

which reduces to

$$(p-1)(p-2) + 2(m-n)(m-p) \gtreqless 0;$$

and this is evidently always true.

A Method of Developing the Perturbative Function of Planetary Motion.

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THE development of the perturbative function in powers of the eccentricities and inclinations has of late been generally regarded as of little value, owing to the complex character of the series to which it leads. It is, in consequence, but little used, even in those cases of nearly circular orbits where it would be most convenient. Still, it is the only development in which the disturbing force is given as an explicit function of all the elements, and is therefore of more interest to the geometer than any other. Moreover, it admits of various simplifications in its application to the numerical problems of celestial mechanics which deserve more attention than they have received, and which may entitle it to a more favorable comparison with other methods than it has been supposed to offer.

The object of the present paper is to exhibit a method of effecting the development in powers of the eccentricities, which seems to me to offer some features of interest, and possibly to contain the germ of some principle which I have not fully grasped, and which may admit of wider and more important applications. I refer especially to the expression of the coefficient of each power of the eccentricity in terms of the coefficients of lower powers, and to the expression of the coefficient of each term involving the perihelia of two planets as the symbolic product of coefficients involving the perihelion of one only. The first of these features was pointed out in a note to the French Academy, found in the *Comptes Rendus,* Vol. LXX. p. 385, the ground of which is covered by the present paper. The second was discovered on the completion of the theory some years later.

One great practical advantage of the process is that it is reduced to a uniform operation of algebraic multiplication, which can be executed by an

unskilled computer, and can be carried to any extent without repeating the previous processes. I see no reason why it might not be possible to express the terms of the moon's longitude by some similar series of operations, and thus greatly simplify the practical problem of the lunar theory. Probably each step would be found to involve the solution of a differential equation, but this equation might be of a very simple character, and only a particular integral would be required.

The present development offers nothing new in the method of taking account of the mutual inclination of the orbits. The development with respect to this element may be made by any method which gives the coefficients as known functions of the radii vectores and the mutual inclination, and the elements under the signs sine and cosine are the multiples of the distances of the two planets from the common node.

The expression to be developed is

$$R = (r^2 - 2rr' \cos V - r'^2)^{-\frac{1}{2}} - \frac{r}{r'^2} \cos V,$$

in which r and r' are the radii vectores of the planets, and V the angle between these radii vectores.

The second term of this expression admits, after the development is effected, of being merged in the first by a simple and well-known modification of certain terms of the first; we shall therefore confine our attention to the first term. The following notation is used:

v, v', the true angular distances of the planets from their common node.
λ, λ', the mean values of v and v'.
y, the mutual inclination of the orbits.
σ, sine $\frac{1}{2}y$.
ρ, ρ', the logarithms of the radii vectores.
υ, υ', the logarithms of the mean distances.
e, e', the eccentricities.
g, g', the mean anomalies.
α, the ratio of the mean distances.

By substituting for $\cos V$ its known value,

$$\cos V = \cos v \cos v' + \sin v \sin v' \cos y$$

or

$$\cos V = (1 - \sigma^2) \cos (v - v') + \sigma^2 \cos (v + v'),$$

and then developing in cosines of multiples of v and v', we shall obtain the terms with which we are to set out. In order not to weary the reader with

what is not essential to the present object, we shall state only those conclusions which form the basis of the new method.

If we suppose the eccentricities to vanish, the value of R can be developed in the form

$$\begin{aligned} R = & \tfrac{1}{2}\Sigma A_i \cos(i\lambda' - i\lambda) \\ & + \Sigma B_i \cos((i+1)\lambda' - (i-1)\lambda) \\ & + \Sigma C_i \cos((i+2)\lambda' - (i-2)\lambda) \\ & + \text{etc.} \end{aligned} \qquad (a)$$

where the index i takes all integral values from positive to negative infinity, and A_i, B_i, C_i, etc. are functions of the mean distances and inclinations which admit of explicit development in powers of σ.

This expression for R may be thrown into the form

$$R = \Sigma_\nu \Sigma_\mu A_{\nu,\mu} \cos(\nu\lambda' + \mu\lambda) \qquad (1)$$

where μ and ν each assume all values from $-\infty$ to $+\infty$, but not independently, being subject to the single restriction that both values must be even, or both odd.

The coefficients A are homogeneous and of the degree -1 in a and a', admitting of being expressed indifferently in either of the forms

$$\frac{1}{a'}\phi\left(\frac{a}{a'}\right) \quad \text{or} \quad \frac{1}{a}\phi\left(\frac{a'}{a}\right).$$

It is in practice more convenient to choose the form in which the fraction $\frac{a}{a'}$ or $\frac{a'}{a}$ shall be less than unity, but, for the purposes of the present investigation, the choice is indifferent.

The original expression for R is of the form

$$R = f(v, v', r, r', \sigma); \qquad (2)$$

and, assuming the eccentricities of both planets to vanish, we have supposed it developed in the form

$$R_0 = f(\lambda, \lambda', a, a', \sigma). \qquad (3)$$

In order to express R as a function of the eccentricities and other elements of the two planets, we must now substitute v, v', r, and r' for λ, λ', a and a' in (3), or, which is the same thing, in (1), the quantities to be substituted being expressed in terms of the eccentricities and mean anomalies.

To continue the process we shall consider R as a function of the logarithms of the radii vectores of the planets, instead of the radii vectores themselves, putting

$$\begin{aligned} \rho &= \log r, \qquad & \rho' &= \log r', \\ \mathfrak{v} &= \log a, & \mathfrak{v}' &= \log a', \end{aligned} \qquad (4)$$

which gives
$$\frac{da}{dv} = a\,; \qquad \frac{dr}{d\rho} = r.$$

It is not necessary to have either A or its derivatives expressed explicitly as a function of v and v', since we have, with respect to any function ϕ of a and a',

$$\frac{d\phi}{dv} = a\frac{d\phi}{da}\,; \qquad \frac{d\phi}{dv'} = a'\frac{d\phi}{da'}\,,$$

a form by which all the derivatives with respect to v and v' may be expressed as functions of a and a'.

All the coefficients of $A_{\mu,\nu}$ being homogeneous and of the degree -1 in a and a', we have

$$\frac{dA}{dv} + \frac{dA}{dv'} + A = 0.$$

We shall hereafter use a symbolic notation, putting D for the operation $\frac{d}{dv}$ and D' for $\frac{d}{dv'}$. All the derivatives of A with respect to v and v' being, like A itself, homogeneous and of the degree -1, we may put, in general,

$$D + D' = -1, \tag{5}$$

and may combine D and D' as if they were multipliers according to the usual rules for such symbols.

To effect the development we require we must, in (2), put

$$\begin{aligned} V &= \lambda + \phi(e, g),\\ \rho &= v + \psi(e, g),\\ V' &= \lambda' + \phi(e', g'),\\ \rho' &= v' + \psi(e', g')\,; \end{aligned} \tag{6}$$

the function ϕ representing the equation of the centre, and ψ the portion of $\log r$ which depends on the eccentricity. From these equations we see that considering R first as a function of V, V', ρ, and ρ', and then as a function of $\lambda, \lambda', v, v', e, e', g$ and g', we shall have for any compound derivative with respect to V and ρ

$$\frac{d^{m+m'+n+n'}R}{dV^m dV'^{m'} d\rho^n d\rho'^{n'}} = \frac{d^{m+m'+n+n'}R}{d\lambda^m d\lambda'^{m'} dv^n dv'^{n'}}\,,$$

that is, *any derivative with respect to V, V', ρ, or ρ' is found by taking the corresponding derivative of the developed function with respect to λ, λ', v, and v'*

Now, supposing that, in (2), V and ρ are replaced by their values in (6), we shall have

$$\begin{aligned}\frac{dR}{de} &= \frac{dR}{dv}\frac{dv}{de} + \frac{dR}{d\rho}\frac{d\rho}{de} \\ &= \frac{dv}{de}\frac{dR}{d\lambda} + \frac{d\rho}{de}\frac{dR}{d\upsilon}. \end{aligned} \tag{7}$$

This equation is the fundamental one in our method of development. By it we express the derivative of R with respect to e in terms of its derivatives with respect to λ and υ. Let us now differentiate this expression with respect to e n times in succession, representing by D_λ the operation $\frac{d}{d\lambda}$. We thus obtain

$$\begin{aligned}\frac{d^{n+1}R}{de^{n+1}} = D_\lambda &\left\{\frac{dv}{de}\frac{d^nR}{de^n} + \binom{n}{1}\frac{d^2v}{de^2}\frac{d^{n-1}R}{de^{n-1}} + \binom{n}{2}\frac{d^3v}{de^3}\frac{d^{n-2}R}{de^{n-2}} + \ldots\right\} \\ + D_\upsilon &\left\{\frac{d\rho}{de}\frac{d^nR}{de^n} + \binom{n}{1}\frac{d^2\rho}{de^2}\frac{d^{n-1}R}{de^{n-1}} + \binom{n}{2}\frac{d^3\rho}{de^3}\frac{d^{n-2}R}{de^{n-2}} + \ldots\right\}\end{aligned} \tag{8}$$

Thus, we have expressed the derivative of any order with respect to e in terms of the derivatives of lower orders, and, by successive substitutions, this derivative will be expressed in terms of derivatives of R with respect to λ and υ.

The coefficient of e^{n+1} in R is found by putting $e = 0$ in (8), and dividing by $1.2.3 \ldots . n+1$. In strictness we should suppose $e = 0$ only after differentiating with respect to λ, but it is evident that the two operations may be interchanged without affecting the result. If we represent this coefficient by $R^{(n+1)}$, and replace the derivatives with respect to e by the corresponding values of this coefficient, namely,

$$\begin{aligned}\frac{dR_0}{de} &= R^{(1)} \\ \frac{d^2R_0}{de^2} &= 2!\,R^{(2)} \\ \frac{d^3R_0}{de^3} &= 3!\,R^{(3)} \\ &\vdots \\ \frac{d^nR_0}{de^n} &= n!\,R^{(n)} \\ \text{etc.} &\quad \text{etc.}\end{aligned}$$

If, also, we represent, as in these last equations, by the subscript zero the operation of putting $e = 0$ after differentiation, we shall have

$$R^{(n+1)} = D_\lambda \left\{ \frac{n!}{(n+1)!} \frac{d\nu_0}{de} R^{(n)} + \binom{n}{1} \frac{(n-1)!}{(n+1)!} \frac{d^2\nu_0}{de^2} R^{(n-1)} + \text{etc.} \ldots \right\}$$
$$+ D_\nu \left\{ \frac{n!}{(n+1)!} \frac{d\rho_0}{de} R^{(n)} + \binom{n}{1} \frac{(n-1)!}{(n+1)!} \frac{d^2\rho_0}{de^2} R^{(n-1)} + \text{etc.} \ldots \right\}.$$

If we represent by ν_n and ρ_n the coefficients of e^n in ν and ρ respectively, we shall have

$$\frac{d^n\nu_0}{de^n} = n!\, \nu_n; \qquad \frac{d^n\rho_0}{de^n} = n!\, \rho_n.$$

Substituting these values in the above equations, we find by simple reductions

$$(n+1) R^{n+1} = D_\lambda \{ \nu_1 R^{(n)} + 2\, \nu_2 R^{(n-1)} + 3\, \nu_3 R^{(n-2)} + \ldots + (n+1) \nu_{n+1} R_0 \}$$
$$+ D_\nu \{ \rho_1 R^{(n)} + 2\, \rho_2 R^{(n-1)} + 3\, \rho_3 R^{(n-2)} + \ldots + (n+1) \rho_{n+1} R_0 \}. \quad (9)$$

Thus, each coefficient is expressed as a linear function of the derivatives of the coefficients of lower orders. We have next to substitute for ν_i and ρ_i their values in terms of g. We have in general

$$i\nu_i = \tfrac{1}{2} \sum_{j=-i}^{j=i} k_j^{(i)} \sin jg; \qquad i\rho_i = \tfrac{1}{2} \sum_{j=-i}^{j=i} h_j^{(i)} \cos jg \qquad (10)$$

from the developments of the elliptic motion, but it is to be remarked that j does not assume all values between the limits $+ i$ and $- i$, but only every alternate value. The special values of k and h to terms of the seventh order are shown in the following scheme. The values for negative values of j are formed from those for positive values by the formulæ

$$k_{-j}^{(i)} = - k_j^{(i)}; \qquad h_{-j}^{(i)} = h_j^{(i)}.$$

In strictness it is not necessary to suppose j negative at all, since the complete value of $4\,\rho_4$, for example, is

$$4\, \rho_4 = \tfrac{1}{2} h_0^{\text{IV}} + h_2^{\text{IV}} \cos 2g + h_4^{\text{IV}} \cos 4g,$$

but it will be found convenient in forming the required functions to suppose j negative in some cases.

Values of h *and* k.

k'_1 $+2$	h'_1 -1	k'_{-1} -2	h'_{-1} -1	0	0	0	0
k''_2 $+\frac{5}{2}$	h''_2 $-\frac{3}{2}$	k''_0 0	h''_0 $+1$	k''_{-2} $-\frac{5}{2}$	h''_{-2} $-\frac{3}{2}$	0	0
k'''_3 $+\frac{13}{4}$	h'''_3 $-\frac{17}{8}$	k'''_1 $-\frac{3}{4}$	h'''_1 $+\frac{9}{8}$	k'''_{-1} $+\frac{3}{4}$	h'''_{-1} $+\frac{9}{8}$	k'''_{-3} $-\frac{13}{4}$	h'''_{-3} $-\frac{17}{8}$
k^{IV}_4 $+\frac{103}{24}$	h^{IV}_4 $-\frac{71}{24}$	k^{IV}_2 $-\frac{11}{6}$	h^{IV}_2 $+\frac{11}{6}$	k^{IV}_0 0	h^{IV}_0 $+\frac{1}{4}$	k^{IV}_{-2} $+\frac{11}{6}$	h^{IV}_{-2} $+\frac{11}{6}$
k^{V}_5 $+\frac{1097}{192}$	h^{V}_5 $-\frac{523}{128}$	k^{V}_3 $-\frac{215}{64}$	h^{V}_3 $+\frac{385}{128}$	k^{V}_1 $+\frac{25}{96}$	h^{V}_1 $+\frac{5}{64}$	k^{V}_{-1} $-\frac{25}{96}$	h^{V}_{-1} $+\frac{5}{64}$
k^{VI}_6 $+\frac{1223}{160}$	h^{VI}_6 $-\frac{899}{160}$	k^{VI}_4 $-\frac{451}{80}$	h^{VI}_4 $+\frac{387}{80}$	k^{VI}_2 $+\frac{17}{32}$	h^{VI}_2 $-\frac{9}{32}$	k^{VI}_0 0	h^{VI}_0 $+\frac{1}{8}$
k^{VII}_7 $+\frac{330911}{32256}$	h^{VII}_7 $-\frac{355081}{46080}$	k^{VII}_5 $-\frac{41699}{4608}$	h^{VII}_5 $+\frac{70273}{9216}$	k^{VII}_3 $+\frac{665}{512}$	h^{VII}_3 $-\frac{5201}{5120}$	k^{VII}_1 $+\frac{749}{4608}$	h^{VII}_1 $+\frac{889}{9216}$

To commence the development, let us take any one term of (1), and for brevity put

$$N = \nu\lambda' + \mu\lambda,$$

and omit writing the indices μ and ν. The term will then be

$$R_0 = R^{(0)} = A \cos N,$$

and its derivatives with respect to λ and v will be

$$\frac{dR^{(0)}}{d\lambda} = -\mu A \sin N,$$

$$\frac{dR^{(0)}}{dv} = DA \cos N.$$

The general formula (9) gives, by putting $n = 0$,

$$R^{(1)} = \nu_1 \frac{dR^{(0)}}{d\lambda} + \rho_1 \frac{dR^{(0)}}{dv},$$

$$= -\mu k_1' A \sin g \sin N + h_1' DA \cos g \cos N,$$

whence

$$2R^{(1)} = (\mu k_1' A \quad + h_1' DA) \cos (N + g)$$
$$+ (-\mu k_1' A + h_1' DA) \cos (N - g).$$

By the repeated application of (9), supposing n successively equal to 1, 2, 3, etc., we shall obtain the successive values of $R^{(n)}$. This process will be facilitated by finding a general formula for passing from $R^{(n)}$ to $R^{(n+1)}$. For this purpose let us put

$$P_j^n, \text{ the coefficient of } \cos (N + jg) \text{ in } R^{(n)},$$

we then have, in the special cases $n = 0$ and $n = 1$,

$$P_0^0 = A,$$
$$P_1^1 = (\mu k_1' \quad + h_1' D) A,$$
$$P_{-1}^1 = (-\mu k_1' + h_1' D) A,$$

and in the general case

$$R^{(n)} = P_n^n \cos (N + ng) + P_{n-2}^n \cos (N + (n-2) g) + \cdots + P_{-n}^n \cos (N - ng),$$

the index j taking each alternate value from $+n$ to $-n$. Differentiating with respect to λ and v, we shall have

$$D_\lambda R^{(n)} = -\mu P_n^n \sin (N + ng) - \mu P_{n-2}^n \sin (N + (n-2) g) - \text{etc.}$$

$$D_v R_{(n)} = DP_n^n \cos (N + ng) + DP_{n-2}^n \cos (N + (n-2) g) + \text{etc.}$$

Putting side by side, and writing in the most condensed form, the pairs of factors which enter into (9), we find them as follows:

$$D_\lambda R^{(n)} = -\mu \sum_{j=-n}^{j=+n} P_j^n \sin (N + jg) \qquad \nu_1 = k_1' \sin g$$

$$D_\lambda R^{(n-1)} = -\mu \sum_{j=-n+1}^{j=n-1} P_j^{n-1} \sin (N + jg) \qquad 2\nu_2 = k_2'' \sin 2g$$

$$D_\lambda R^{(n-2)} = -\mu \sum_{j=-n+2}^{j=n-2} P_j^{n-2} \sin (N + jg) \qquad 3\nu_3 = k_3''' \sin 3g + k_1''' \sin g$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$D_\lambda R^{(0)} = -\mu P_0^0 \sin N \qquad (n+1)\nu_{n+1} = \tfrac{1}{2} \sum_{i=-n-1}^{i=n+1} k_i^{(n+1)} \sin ig$$

$$
\begin{array}{llr}
D_0R^{(n)} & = D\sum\limits_{j=-n}^{j=n} P_j^n \cos(N+jg) & \rho_1 = h'_1 \cos g \\
D_0R^{(n-1)} & = D\sum\limits_{j=-n+1}^{j=n-1} P_j^{n-1} \cos(N+jg) & 2\,\rho_2 = h''_2 \cos 2g + \tfrac{1}{2} h''_0 \\
D_0R^{(n-2)} & = D\sum\limits_{j=-n+2}^{j=n-2} P_j^{n-2} \cos(N+jg) & 3\,\rho_3 = h'''_3 \cos 3g + h'''_1 \cos g \\
\vdots & \quad\vdots & \vdots \qquad \vdots \\
D_0R^{(0)} & = DP_0^0 \cos N & (n+1)\,\rho_{n+1} = \tfrac{1}{2}\sum\limits_{i=-n-1}^{i=n+1} h_i^{(n+1)} \cos ig.
\end{array}
$$

If we form the products of these quantities according to the formula (9), and compare the coefficients of the several angles, we find

$$
\begin{aligned}
2\,(n+1)\,P_{n+1}^{n+1} = {} & (\mu k'_1 + h'_1 D)\,P_n^n \\
& + (\mu k''_2 + h''_2 D)\,P_{n-1}^{n-1} \\
& + (\mu k'''_3 + h'''_3 D)\,P_{n-2}^{n-2} \\
& + \cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \\
& + (\mu k_{n+1}^{(n+1)} + h_{n+1}^{(n+1)} D)\,P_0^0
\end{aligned}
$$

$$
\begin{array}{ll}
2\,(n+1)\,P_{n-1}^{n+1} = (\mu k'_1 + h'_1 D)\,P_{n-2}^n & + (\mu k'_{-1} + h'_{-1} D)\,P_n^n \\
\qquad + (\mu k''_2 + h''_2 D)\,P_{n-3}^{n-1} & + \qquad\qquad h''_0 D P_{n-1}^{n-1} \\
\qquad + \text{etc.} & + (\mu k'''_1 + h'''_1 D)\,P_{n-2}^{n-2} \qquad\qquad (11) \\
\qquad + (\mu k_n^{(n)} + h_n^{(n)} D)\,P_{-1}^1 & + \text{etc} \\
 & + (\mu k_{n-1}^{(n+1)} + h_{n-1}^{(n+1)} D)\,P_0^0
\end{array}
$$

$$
\begin{aligned}
2(n+1)\,P_{n-3}^{n+1} = {} & (\mu k'_1 + h'_1 D)\,P_{n-4}^n + (\mu k'_{-1} + h'_{-1} D)\,P_{n-2}^n \\
& + (\mu k''_2 + h''_2 D)\,P_{n-5}^{n-1} + \qquad h''_0 D P_{n-3}^{n-1} + (\mu k''_{-2} + h''_{-2} D)\,P_{n-1}^{n-1} \\
& + (\mu k'''_3 + h'''_3 D)\,P_{n-6}^{n-2} + (\mu k'''_1 + h'''_1 D)\,P_{n-4}^{n-2} + (\mu k'''_{-1} + h'''_{-1} D)\,P_{n-2}^{n-2} \\
& + \qquad\qquad + \qquad\qquad + \\
& \;\vdots \qquad\qquad\;\; \vdots \qquad\qquad\;\; \vdots \\
& + (\mu k_{n-1}^{(n-1)} + h_{n-1}^{(n-1)})\,P_{-2}^2 + (\mu k_{n-2}^{(n)} + h_{n-2}^{(n)} D)\,P'_{-1} + (\mu k_{n-3}^{(n+1)} + h_{n-3}^{(n+1)} D)\,P_0^0
\end{aligned}
$$

To show the law of progression of the several classes of terms, I have purposely written some of the coefficients k and h with negative indices instead of using the corresponding positive ones, which, in the actual development, will be

substituted for them. The law can be seen by induction and comparison without a full statement of it. As we diminish the value of the lower exponent of P by successive steps, two units at a time, the number of columns of products increases by one at each step, but the number of products in each column diminishes in consequence of one of the factors vanishing whenever a lower exponent in h, k, or P exceeds the upper one in absolute value. Supposing the successive values of P_j^{n+1} to be written in this form with continually diminishing values of j until we reach the value $j = -(n+1)$, to this value would correspond $n+2$ columns, of which, however, the first would entirely vanish, as would every term of the remaining ones, except the first. Placing these terms in a column, they will be as follows:—

$$
\begin{array}{ll}
P^{n+1}_{-n-1} = (\mu k'_{-1} + h'_{-1}D)\, P^n_{-n} & = (-\mu k'_1 + h'_1 D)\, P^n_{-n} \\
\quad + (\mu k''_{-2} + h''_{-2}D)\, P^{n-1}_{-n+1} & \quad + (-\mu k''_2 + h''_2 D)\, P^{n-1}_{-n+1} \\
\quad + (\mu k'''_{-3} + h'''_{-3}D)\, P^{n-2}_{-n+2} & \quad + (-\mu k'''_3 + h'''_3 D)\, P^{n-2}_{-n+2} \\
\quad + & \quad + \\
\quad \cdot & \quad \cdot \\
\quad \cdot & \quad \cdot \\
\quad \cdot & \quad \cdot \\
\quad + (\mu k^{(n+1)}_{-n-1} + h^{(n+1)}_{-n-1}\, D)\, P^0_0 & \quad + (-\mu k^{(n+1)}_{n+1} + h^{(n+1)}_{n+1} D)\, P^0_0
\end{array}
$$

This is a particular case of the general law of formation of the values of P when j is negative, which law may be expressed as follows:—

Each value of P *with* j *negative may be formed from the corresponding value for* j *positive by changing the sign of* μ, *and of the lower exponents in all the values of* P *which enter into it.*

We have next to consider the development with respect to the powers of e'. What we have hitherto done has been to take

$$R = f(v, \rho, v', \rho')$$

and substituting

$$v = \lambda + \phi, (e, g);$$

$$\rho = \upsilon + \psi, (e, g),$$

to find the coefficients $R^{(n)}$ of the development with respect to e only. We have written λ' and υ' instead of v and ρ in giving the values of the coefficients. Now, when we replace λ' and υ' by v' and ρ', expressed in terms of the elements, each coefficient $R^{(n)}$ will become a function of v' and ρ', and hence of e', g', etc. We shall now proceed to develop each of these coefficients in powers of e' by the same process which was followed in developing R in powers of e. Let

$$P_j^n \cos(\nu\lambda' + \mu\lambda + jg)$$

be any term of $R^{(n)}$, and, having replaced λ' by V' and ϖ' by ρ' in this term, let us suppose

$$V' = \lambda' + \phi(e', g'),$$
$$\rho' = \varpi' + \psi(e', g');$$

and let us put

$R^{n,n'}$, the coefficient of $e'^{n'}$ in this development.

We shall then have, as in (9),

$$(n'+1)\,R^{n,n'+1} = V'_1 D_{\lambda'} R^{n,n'} + 2\,V'_2 D_{\lambda'} R^{n,n'-1} + \ldots + (n'+1)\,V'_{n'+1} R^{n,0}$$
$$+ \rho'_1 D' R^{n,n'} + 2\,\rho'_2 D' R^{n,n'-1} + \ldots + (n'+1)\,\rho'_{n'+1} R^{n,0}. \quad (12)$$

The expressions for V' and ρ' will be the same as those for V and ρ in (10), except that g' is to be substituted for g. We may now investigate the general law of development in the same way as before. Putting, for brevity,

$$N' = \mu\lambda + \nu\lambda' + jg = N + jg,$$

so that

$$P^n_j \cos N'$$

is any term of $R^{(n)}$ or, which is the same thing, of $R^{n,0}$, let us represent the corresponding terms in $R^{n,n'}$ by

$$R^{n,n'} = P^{n;\,n'}_{j,\,n'} \cos(N' + n'g') + P^{n;\,n'}_{j,\,n'-2} \cos(N' + (n'-2)g') + \ldots + P^{n;\,n'}_{j,\,-n'} \cos(N' - n'g').$$

Then, proceeding as before, we shall find

$$\begin{aligned} 2(n'+1)\,P^{n;\,n'+1}_{j,\,-n'+1} = {} & (\nu k'_1 + h'_1 D')\,P^{n;\,n'}_{j,\,n'} \\ & + (\nu k''_2 + h''_2 D')\,P^{n;\,n'-1}_{j,\,n'-1} \\ & + \ldots\ldots\ldots \\ & + (\nu k^{(n'+1)}_{n'+1} + h^{(n'+1)}_{n'+1} D')\,P^{n;\,0}_{j,\,0}. \end{aligned} \quad (13)$$

All the other coefficients can be formed from the corresponding ones of (11) by writing

ν for μ,
D' for D,
P^n_j for P,
n' for n.

We have now an important law of this development of R to bring out. First, in the equations (11), by supposing in succession $n = 1$, $n = 2$, $n = 3$, etc., and by continually substituting in each set of terms the values of those of a lower order, we shall finally express all the values of P^n_j in terms of $P^0_0 = A$ and its successive

derivatives with respect to ν. Moreover, the operation of forming these derivatives being always linear, we can combine all the operations represented by the symbols $\mu k + hD$ as if D represented a coefficient; that is, having the quantity

$$(\mu k + hD)(\mu k' + h'D) \ldots . (\mu k_n + h_n D) P_0^0$$

we can multiply these several symbols as if D were a coefficient. By this operation, putting A for P_0^0, we shall finally obtain an expression of the form

$$P_j^n = \Pi_j^n A,$$

in which Π_j^n represents an entire function of μ and D of the degree n.

Secondly, by treating the equations represented by (13) in the same manner, we shall be able to represent each value of $P_{j\,;\,j'}^{n\,;\,n'}$ in the form

$$P_{j\,;\,j'}^{n\,;\,n'} = \Pi_{j'}^{n'} (P_{j\,;\,0}^{n\,;\,0} = P_j^n)$$

in which $\Pi_{j'}^{n'}$ represents an entire function of ν and D' of the degree n'. Substituting for P_j^n its value just given, we shall have

$$P_{j\,;\,j'}^{n\,;\,n'} = \Pi_{j'}^{n'} \Pi_j^n A \tag{14}$$

in which the two symbols can be combined by the rule of multiplication. *It thus appears that when we have found the development in powers of* e *for* $e' = 0$, *and that in* e' *for* $e = 0$, *we have solved the whole problem, and the terms multiplied by any product of a power of* e *by a power of* e' *can then be found by a symbolic multiplication.*

It may not be amiss to recapitulate the result which we have reached. Suppose that we develop R in powers of e on the supposition $e' = 0$, and that any term of this development is represented in the form

$$R = e^n \Pi_j^n A \cos (N + jg),$$

A being a function of the mean distances, Π_j^n, an operating symbol, and N a function of λ and λ' which does not contain g. Suppose, next, that we develop R in powers of e', putting $e = 0$, and that any term of this development is represented by

$$R = e'^{n'} \Pi_{j'}^{n'} A \cos (N + j'g'),$$

then the coefficient of $\cos (N + jg + j'g')$ in the complete development will be represented by

$$e^n e'^{n'} \Pi_j^n \Pi_{j'}^{n'} A. \tag{15}$$

To proceed to the actual development, it is necessary to form the symbolic factors represented by Π_j^n and $\Pi_{j'}^{n'}$. This we may do from the forms (11) by substituting for h and k their numerical values, and substituting the symbol

Π^i_j for P^i_j, remembering that $\Pi^0_0 = 1$. We thus have, for each successive value of n,

$$
\begin{aligned}
2(n+1)\Pi^{n+1}_{n+1} = {} & (2\mu - D)\Pi^n_n \\
& + \left(\frac{5}{2}\mu - \frac{3}{2}D\right)\Pi^{n-1}_{n-1} \\
& + \left(\frac{13}{4}\mu - \frac{17}{8}D\right)\Pi^{n-2}_{n-2} \\
& + \left(\frac{103}{24}\mu - \frac{71}{24}D\right)\Pi^{n-3}_{n-3} \\
& + \left(\frac{1097}{192}\mu - \frac{523}{128}D\right)\Pi^{n-4}_{n-4} \\
& + \left(\frac{1223}{160}\mu - \frac{899}{160}D\right)\Pi^{n-5}_{n-5} \\
& + \text{etc.}
\end{aligned}
$$

the series terminating with Π^0_0.

$$
\begin{aligned}
2(n+1)\Pi^{n+1}_{n-1} = {} & (2\mu - D)\Pi^n_{n-2} && + (-2\mu - D)\Pi^n_n \\
& + \left(\frac{5}{2}\mu - \frac{3}{2}D\right)\Pi^{n-1}_{n-3} && + D\Pi^{n-1}_{n-1} \\
& + \left(\frac{13}{4}\mu - \frac{17}{8}D\right)\Pi^{n-2}_{n-4} && + \left(-\frac{3}{4}\mu + \frac{9}{8}D\right)\Pi^{n-2}_{n-2} \\
& + \text{etc.} && + \text{etc.} \\
& + (k^{(n)}_n\mu + h^{(n)}_n D)\Pi'_{-1} && + (k^{(n+1)}_{n-1}\mu + h^{(n+1)}_{n-1}D)\Pi^0_0
\end{aligned}
$$

$$
\begin{aligned}
2(n+1)\Pi^{n+1}_{n-3} = {} & (2\mu - D)\Pi^n_{n-4} && + (-2\mu - D)\Pi^n_{n-2} \\
& + \left(\frac{5}{2}\mu - \frac{3}{2}D\right)\Pi^{n-1}_{n-5} && + D\Pi^{n-1}_{n-3} + \left(-\frac{5}{2}\mu - \frac{3}{2}D\right)\Pi^{n-1}_{n-1} \\
& + \left(\frac{13}{4}\mu - \frac{17}{8}D\right)\Pi^{n-2}_{n-6} && + \left(-\frac{3}{4}\mu + \frac{9}{8}D\right)\Pi^{n-2}_{n-4} + \left(\frac{3}{4}\mu + \frac{9}{8}D\right)\Pi^{n-2}_{n-2} \\
& + \text{etc.} && + \text{etc.} \qquad\qquad + \text{etc.} \\
& + (k^{(n-1)}_{n-1}\mu + h^{(n-1)}_{n-1}D)\Pi^2_{-2} && + (k^{(n)}_{n-2}\mu + h^{(n)}_{n-2}D)\Pi^1_{-1} + (k^{(n+1)}_{n-3}\mu + h^{(n+1)}_{n-3}D)\Pi^0_0
\end{aligned}
$$

$$
\begin{aligned}
2(n+1)\Pi^{n+1}_{n-5} = {} & (2\mu - D)\Pi^n_{n-6} && + (-2\mu - D)\Pi^n_{n-4} \\
& + \left(\frac{5}{2}\mu - \frac{3}{2}D\right)\Pi^{n-1}_{n-7} && + D\Pi^{n-1}_{n-5} + \left(-\frac{5}{2}\mu - \frac{3}{2}D\right)\Pi^{n-1}_{n-3} \\
& + \left(\frac{13}{4}\mu - \frac{17}{8}D\right)\Pi^{n-2}_{n-8} && + \left(-\frac{3}{4}\mu + \frac{9}{8}D\right)\Pi^{n-2}_{n-6} + \left(\frac{3}{4}\mu + \frac{9}{8}D\right)\Pi^{n-2}_{n-4} \\
& && + \left(-\frac{13}{4}\mu - \frac{17}{8}D\right)\Pi^{n-2}_{n-2} \\
& + \text{etc.} && + \text{etc.} \qquad\qquad + \text{etc.} \\
& + (k^{(n-2)}_{n-2}\mu + h^{(n-2)}_{n-2}D)\Pi^3_{-3} && + (k^{(n-1)}_{n-3}\mu + h^{(n-1)}_{n-3}D)\Pi^2_{-2} + (k^{(n)}_{n-4}\mu + h^{(n)}_{n-4}D)\Pi^1_{-1} \\
& && + (k^{(n+1)}_{n-5}\mu + h^{(n+1)}_{n-5}D)\Pi^0_0
\end{aligned}
$$

$$
\begin{aligned}
2(n+1)\,\Pi^{n+1}_{-n-1} = {} & (-2\mu - D)\,\Pi^{n}_{-n} \\
& + \left(-\tfrac{5}{2}\mu - \tfrac{3}{2}D\right)\Pi^{n-1}_{-n+1} \\
& + \left(-\tfrac{13}{4}\mu - \tfrac{17}{8}D\right)\Pi^{n-2}_{-n+2} \\
& + \text{etc.} \\
& + (-k^{(n+1)}_{n+1}\mu + h^{(n+1)}_{n+1}D)\,\Pi^{0}_{0}
\end{aligned}
$$

$$
\begin{aligned}
2(n+1)\,\Pi^{n+1}_{-n+1} = {} & (-2\mu - D)\,\Pi^{n}_{-n+2} && + (2\mu - D)\,\Pi^{n}_{-n} \\
& + \left(-\tfrac{5}{2}\mu - \tfrac{3}{2}D\right)\Pi^{n-1}_{-n+3} && + D\Pi^{n-1}_{-n+1} \\
& + \left(-\tfrac{13}{4}\mu - \tfrac{17}{8}D\right)\Pi^{n-2}_{-n+4} && + \left(\tfrac{3}{4}\mu + \tfrac{9}{8}D\right)\Pi^{n-2}_{-n+2} \\
& + \text{etc.} && + \text{etc.} \\
& + (-k^{(n)}_{n}\mu + h^{(n)}_{n}D)\,\Pi^{1}_{1} && + (-k^{(n+1)}_{n-1}\mu + h^{(n+1)}_{n-1}D)\,\Pi^{0}_{0}
\end{aligned}
$$

The actual computation of Π^n_j for negative values of j is not necessary, since its values may be obtained from those for positive j by changing the sign of μ. These numerical coefficients may be continued to any extent by means of the scheme of values of h and k already given, it being remarked that the successive columns appear in the same order as in the scheme.

We are now ready to proceed with the actual computation of the functions $P^{n}_{j}{}^{;}{}^{n'}_{j'}$, or, which is the same thing, of the symbolic functions $\Pi^{n}_{j}{}^{;}{}^{n'}_{j'}$ which express the values of $P^{n}_{j}{}^{;}{}^{n'}_{j'}$ when considered as operators on the functions A_i, B_i, etc. We shall give only a few of these functions for the purpose of illustrating the method.

We consider, firstly, the general values of $\Pi^{n}_{j}{}^{;}{}^{0}_{0}$ and of $\Pi^{0}_{0}{}^{;}{}^{n'}_{j'}$ which arise when, in the general term of R for circular orbits,

$$R = A\cos(\nu\lambda' + \mu\lambda),$$

we substitute for the mean values of the radii vectores and longitudes those which correspond to the elliptic motion. The functions of Π for values of j and j' to the fourth order, inclusive, are then found to be as follows. It is not necessary to write the functions for negative values of j or j', because they are found from the corresponding ones for positive values by simply changing the sign of μ and D, but a few are given for perspicuity.

$$\Pi_0^0 = 1$$

$$\Pi_1^1 = \mu - \frac{1}{2} D$$

$$\Pi_{-1}^1 = - \mu - \frac{1}{2} D$$

$$\Pi_2^2 = \frac{1}{2}\mu^2 + \frac{5}{8}\mu + \left(-\frac{1}{2}\mu - \frac{3}{8}\right) D + \frac{1}{8} D^2$$

$$\Pi_0^2 = -\mu^2 \qquad + \frac{1}{4}\, D + \frac{1}{4} D^2$$

$$\Pi_{-2}^2 = \frac{1}{2}\mu^2 - \frac{5}{8}\mu + \left(\frac{1}{2}\mu - \frac{3}{8}\right) D + \frac{1}{8} D^2$$

$$\Pi_3^3 = \frac{1}{6}\mu^3 + \frac{5}{8}\mu^2 + \frac{13}{24}\mu + \left(-\frac{1}{4}\mu^2 - \frac{11}{16}\mu - \frac{17}{48}\right) D + \left(\frac{1}{8}\mu + \frac{3}{16}\right) D^2 - \frac{1}{48} D^3$$

$$\Pi_1^3 = -\frac{1}{2}\mu^3 - \frac{5}{8}\mu^2 - \frac{1}{8}\mu + \left(\frac{1}{4}\mu^2 + \frac{5}{16}\mu + \frac{3}{16}\right) D + \left(\frac{1}{8}\mu + \frac{1}{16}\right) D^2 - \frac{1}{16} D^3.$$

$$\Pi_{-1}^3 = \frac{1}{2}\mu^3 - \frac{5}{8}\mu^2 + \frac{1}{8}\mu + \left(\frac{1}{4}\mu^2 - \frac{5}{16}\mu + \frac{3}{16}\right) D + \left(-\frac{1}{8}\mu + \frac{1}{16}\right) D^2 - \frac{1}{16} D^3.$$

$$\Pi_{-3}^3 = -\frac{1}{6}\mu^3 + \frac{5}{8}\mu^2 - \frac{13}{24}\mu + \left(-\frac{1}{4}\mu^2 + \frac{11}{16}\mu - \frac{17}{48}\right) D + \left(-\frac{1}{8}\mu + \frac{3}{16}\right) D^2 - \frac{1}{48} D^3$$

$$\Pi_4^4 = \frac{1}{24}\mu^4 + \frac{5}{16}\mu^3 + \frac{283}{384}\mu^2 + \frac{103}{192}\mu + \left(-\frac{1}{12}\mu^3 - \frac{1}{2}\mu^2 - \frac{51}{64}\mu - \frac{71}{192}\right) D$$
$$+ \left(\frac{1}{16}\mu^2 + \frac{17}{64}\mu + \frac{95}{384}\right) D^2 + \left(-\frac{1}{48}\mu - \frac{3}{64}\right) D^3 + \frac{1}{384} D^4$$

$$\Pi_2^4 = -\frac{1}{6}\mu^4 - \frac{5}{8}\mu^3 - \frac{2}{3}\mu^2 - \frac{11}{48}\mu + \left(\frac{1}{6}\mu^3 + \frac{1}{2}\mu^2 + \frac{47}{96}\mu + \frac{11}{48}\right) D$$
$$+ \left(\frac{1}{32}\mu - \frac{1}{96}\right) D^2 + \left(-\frac{1}{24}\mu - \frac{1}{16}\right) D^3 + \frac{1}{96} D^4$$

$$\Pi_0^4 = \frac{1}{4}\mu^4 - \frac{9}{64}\mu^2 + \frac{1}{32} D + \left(-\frac{1}{8}\mu^2 - \frac{1}{64}\right) D^2 - \frac{1}{32} D^3 + \frac{1}{64} D^4$$

The values of Π' being formed, as already shown, by simply changing μ into ν and D into D', it is unnecessary to write them as functions of D'. As it will be convenient to have but one form of derivative, they should be transformed, the symbol D' being replaced by D by writing

$$D' = -(1 + D).$$

We thus find

$$\Pi^{0;1}_{0,1} = \nu + \tfrac{1}{2} + \tfrac{1}{2} D$$

$$\Pi^{0;2}_{0,2} = \tfrac{1}{2}\nu^2 + \tfrac{9}{8}\nu + \tfrac{1}{2} + \left(\tfrac{1}{2}\nu + \tfrac{5}{8}\right) D + \tfrac{1}{8} D^2$$

$$\Pi^{0;2}_{0,0} = -\nu^2 \qquad\qquad + \tfrac{1}{4} D + \tfrac{1}{4} D^2$$

$$\Pi^{0;3}_{0,3} = \tfrac{1}{6}\nu^3 + \tfrac{7}{8}\nu^2 + \tfrac{65}{48}\nu + \tfrac{9}{16} + \left(\tfrac{1}{4}\nu^2 + \tfrac{15}{16}\nu + \tfrac{19}{24}\right) D + \left(\tfrac{1}{8}\nu + \tfrac{1}{4}\right) D^2 + \tfrac{1}{48} D^3$$

$$\Pi^{0;3}_{0,1} = -\tfrac{1}{2}\nu^3 - \tfrac{7}{8}\nu^2 - \tfrac{5}{16}\nu - \tfrac{1}{16} + \left(-\tfrac{1}{4}\nu^2 - \tfrac{1}{16}\nu + \tfrac{1}{8}\right) D + \left(\tfrac{1}{8}\nu + \tfrac{1}{4}\right) D^2 + \tfrac{1}{16} D^3$$

$$\Pi^{0;4}_{0,4} = \tfrac{1}{24}\nu^4 + \tfrac{19}{48}\nu^3 + \tfrac{499}{384}\nu^2 + \tfrac{323}{192}\nu + \tfrac{2}{3} + \left(\tfrac{1}{12}\nu^3 + \tfrac{5}{8}\nu^2 + \tfrac{93}{64}\nu + \tfrac{65}{64}\right) D$$
$$+ \left(\tfrac{1}{16}\nu^2 + \tfrac{21}{64}\nu + \tfrac{155}{384}\right) D^2 + \left(\tfrac{1}{48}\nu + \tfrac{11}{192}\right) D^3 + \tfrac{1}{384} D^4$$

$$\Pi^{0;4}_{0,2} = -\tfrac{1}{6}\nu^4 - \tfrac{19}{24}\nu^3 - \tfrac{7}{6}\nu^2 - \tfrac{31}{48}\nu - \tfrac{1}{6} + \left(-\tfrac{1}{6}\nu^3 - \tfrac{1}{2}\nu^2 - \tfrac{29}{96}\nu - \tfrac{1}{48}\right) D$$
$$+ \left(\tfrac{5}{32}\nu + \tfrac{23}{96}\right) D^2 + \left(\tfrac{1}{24}\nu + \tfrac{5}{48}\right) D^3 + \tfrac{1}{96} D^4$$

$$\Pi^{0;4}_{0,0} = \tfrac{1}{4}\nu^4 - \tfrac{17}{64}\nu^2 + \left(-\tfrac{1}{4}\nu^2 + \tfrac{3}{32}\right) D + \left(-\tfrac{1}{8}\nu^2 + \tfrac{11}{64}\right) D^2 + \tfrac{3}{32} D^3 + \tfrac{1}{64} D^4.$$

The symbolic products

$$\Pi^{n;n'}_{j;j'} = \Pi^{n}_{j} \times \Pi^{0;n'}_{0;j'}$$

are functions of both the indices μ and ν. Instead of using them in their general form, it is more convenient to apply them to the separate terms of the development (a). To develop the first term, we put

$$\mu = -i\,;\ \nu = i.$$

For the second,

$$\mu = -i + 1\,;\ \nu = i + 1, \text{ etc.}$$

To show the forms to which we are thus led, the following exhibit of the first four orders of terms arising from the first term of (a) is presented. The symbol (i) indicates that in Π, μ is changed to $-i$ and ν to i.

Developed Value of R.

$$
\begin{aligned}
R = \; & \tfrac{1}{2}\, \{1 + e^2\Pi_0^2 + e'^2\Pi_{0,0}^{0;2} + e^4\Pi_0^4 + e^2e'^2\Pi_{0,0}^{2;2} + e'^4\Pi_{0,0}^{0;4} + \text{etc.}\}^{(0)} A_0 \\
& + \sum_{i=1}^{i=\infty} \\
& \{1 + e^2\Pi_0^2 + e'^2\Pi_{0,0}^{0;2} + e^4\Pi_{0,0}^{4;0} + e^2e'^2\Pi_{0,0}^{2;2} + e'^4\Pi_{0,0}^{0;4} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda) \\
& + \sum_{i=-\infty}^{i=+\infty} \\
+ e \; & \{\Pi_{1,0}^{1;0} + e^2\Pi_{1,0}^{3;0} + e'^2\Pi_{1,0}^{1;2} + e^4\Pi_{1,0}^{5;0} + e^2e'^2\Pi_{1,0}^{3;2} + e'^4\Pi_{1,0}^{1;4} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda + g) \\
+ e' \; & \{\Pi_{0,1}^{0;1} + e^2\Pi_{0,1}^{2;1} + e'^2\Pi_{0,1}^{0;3} + e^4\Pi_{0,1}^{4;1} + e^2e'^2\Pi_{0,1}^{2;3} + e'^4\Pi_{0,1}^{0;5} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda + g') \\
+ e^2 \; & \{\Pi_{2,0}^{2;0} + e^2\Pi_{2,0}^{4;0} + e'^2\Pi_{2,0}^{2;2} + e^4\Pi_{2,0}^{6;0} + e^2e'^2\Pi_{2,0}^{4;2} + e'^4\Pi_{2,0}^{2;4} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda + 2g) \\
+ ee' \; & \{\Pi_{1,1}^{1;1} + e^2\Pi_{1,1}^{3;1} + e'^2\Pi_{1,1}^{1;3} + e^4\Pi_{1,1}^{5;1} + e^2e'^2\Pi_{1,1}^{3;3} + e'^4\Pi_{1,1}^{1;5} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda + g' + g) \\
+ e'^2 \; & \{\Pi_{0,2}^{0;2} + e^2\Pi_{0,2}^{2;2} + e'^2\Pi_{0,2}^{0;4} + e^4\Pi_{0,2}^{4;2} + e^2e'^2\Pi_{0,2}^{2;4} + e'^4\Pi_{0,2}^{0;6} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda + 2g') \\
+ e^3 \; & \{\Pi_{3,0}^{3;0} + e^2\Pi_{3,0}^{5;0} + e'^2\Pi_{3,0}^{3;2} + e^4\Pi_{3,0}^{7;0} + e^2e'^2\Pi_{3,0}^{5;2} + e'^4\Pi_{3,0}^{3;4} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda + 3g) \\
+ e^2e' \; & \{\Pi_{2,1}^{2;1} + e^2\Pi_{2,1}^{4;1} + e'^2\Pi_{2,1}^{2;3} + e^4\Pi_{2,1}^{6;1} + e^2e'^2\Pi_{2,1}^{4;3} + e'^4\Pi_{2,1}^{2;5} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda + g' + 2g) \\
+ ee'^2 \; & \{\Pi_{1,2}^{1;2} + e^2\Pi_{1,2}^{3;2} + e'^2\Pi_{1,2}^{1;4} + e^4\Pi_{1,2}^{5;2} + e^2e'^2\Pi_{1,2}^{3;4} + e'^4\Pi_{1,2}^{1;6} + \text{etc.}\}^{(i)} A^i \cos(i\lambda' - i\lambda + 2g' + g) \\
+ e^{\;3} \; & \{\Pi_{0,3}^{0;3} + e^2\Pi_{0,3}^{2;3} + e'^2\Pi_{0,3}^{0;5} + e^4\Pi_{0,3}^{4;3} + e^2e'^2\Pi_{0,3}^{2;5} + e'^4\Pi_{0,3}^{0;7} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda + 3g') \\
+ e^4 \; & \{\Pi_{4,0}^{4;0} + e^2\Pi_{4,0}^{6;0} + e'^2\Pi_{4,0}^{4;2} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda + 4g) \\
+ e^3e' \; & \{\Pi_{3,1}^{3;1} + e^2\Pi_{3,1}^{5;1} + e'^2\Pi_{3,1}^{3;3} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda + g' + 3g) \\
+ e^2e'^2 & \{\Pi_{2,2}^{2;2} + e^2\Pi_{2,2}^{4;2} + e'^2\Pi_{2,2}^{2;4} + \text{etc}\}^{(i)} A_i \cos(i\lambda' - i\lambda + 2g' + 2g) \\
+ ee'^3 \; & \{\Pi_{1,3}^{1;3} + e^2\Pi_{1,3}^{3;3} + e'^2\Pi_{1,3}^{1;5} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda + 3g' + g) \\
+ e'^4 \; & \{\Pi_{0,4}^{0;4} + e^2\Pi_{0,4}^{2;4} + e'^2\Pi_{0,4}^{0;6} + \text{etc.}\}^{(i)} A_i \cos(i\lambda' - i\lambda + 4g')
\end{aligned}
$$

This development is not directly comparable with those hitherto executed, because we use the successive logarithmic derivatives D, instead of the derivatives with respect to α, the ratio of the mean distances. The two classes of derivatives are, however, connected by a simple linear relation which makes it easy to pass from one to the other.

On De Morgan's Extension of the Algebraic Processes.

By Miss Christine Ladd,

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The algebra which I am about to consider is a symbolic algebra; that is to say, none of the processes or symbols used will have any necessary meaning. Processes will be indicated by symbols whose definitions consist in their laws of combination, and the interpretation to be given of final results will depend on what particular meaning may be given to the starting symbols. The word addition, for instance, need be nothing more than a "sound void of sense," used to indicate the process denoted by $+$.

The principles to which every symbolical algebra must be subjected are the following: *

1. A symbol is the representative of one process and only one.

2. Any number of processes may be looked at in their united effect as one process and represented by one symbol.

3. Every process by which we can pass from one object of contemplation to another involves a second by which we can reinstate the first object in its position; or every direct process has another which is its inverse.

The only necessary symbols are the following: $=$, $+$, $-$, log. All the processes of algebra can be expressed in terms of these symbols. The laws of combination to which they are subject are these:

1. $a = b$ means that wherever a occurs b may be substituted for it.

2. $+$ and $-$ are symbols of inverse operations; whatever one does, the other undoes; as, $a + b - b = a$. They are both subject to the following laws of (1) commutation and (2) distribution:

$$a + b - c = b - c + a = b + a - c \tag{1}$$

$$(a + b) + c = a + (b + c) \tag{2}$$

---

* De Morgan, Camb. Phil. Trans., VII. 176.

Both principles are expressed at once in the equation

$$a + (b - c) = (a - c) + b. \tag{3}$$

The word distribution is here used in a sense more general than the common one. $(a + b)\,c = ac + bc$ is commonly said to express *the* principle of distribution, but it is, in fact, only a particular case of it; namely, it gives the rule for distributing c over a and b when c is a multiplier and a and b are connected by the sign of addition. The index-law and the binomial theorem are rules for the distribution of c over a and b in the expressions $(ab)^c$ and $(a + b)^c$. In its most general sense, the law of distribution simply determines f, f_1, f_2 in

$$\phi\,[\psi\,(a, b), c] = f\,[f_1\,(a, c), f_2\,(b, c)]$$

when ϕ and ψ are given. (2) expresses the fact that if $\phi = \psi =$ *addition of*, then we must have $f_1\,(a, c) = a$, $f_2\,(b, c) = b + c$, and $f = f_1 + f_2$. The c, in fact, disappears from either f_1 or f_2, but so in the expansion of $(a + b)^c$, when c is negative or fractional, c is not explicitly applied to both a and b.

The expression *is a brother or sister of* obeys the above laws of commutation and distribution, and is therefore a possible meaning of $+$; but *is a father of* is not a possible meaning. The interpretation of (3) might be, "The statement that a is a brother or sister of (b, who is not a brother or sister of c) is equivalent to the statement that (a, who is not a brother or sister of c) is a brother or sister of b."

3. The symbol log is defined by the equation

$$\log\,(a + a + \cdots \text{ to } b \text{ terms}) = \log a + \log b. \tag{4}$$

It may be shown hereafter that the logarithm of a is the only function of a which possesses this property, and reason may be given for adopting the natural base; but for the present it is not necessary to attribute any meaning to the symbol *log* other than that contained in equation (4). This is, in fact, the way in which the idea of the logarithm was first introduced by Napier.

All the processes of algebra might be expressed in terms of these symbols alone, but for the sake of simplicity of notation other symbols, which are abbreviations of combinations of these, may be employed. We shall write for $\log\,(\log a)$, $\log^2 a$; for $\log\log\log a$, $\log^3 a$; etc. It will then be necessary to express the square of $\log a$ by $(\log a)^2$. By the definition already given of $+$ and $-$, we have $\log^{-n}(\log^{+n} a) = a$; or $\log^{-n}$ is a process whose effect is to annul that of $\log^n$. The expression $\log^{-n} a$ may be read *the inverse* n*th* *log of* a, or *the quantity whose* n*th* *log is* a. It cannot in general be taken for granted that an inverse function is a

single-valued function. We have, for instance, $\frac{d}{dx}\int f(x) = f(x)$, but $\int \frac{d}{dx} f(x)$ is equal to $f(x)$ only for one definite value of the constant of integration. When log has been identified with the ordinary logarithm, it will appear that it is a periodic function and that $\log^{-1}$ is single-valued, and hence that $\log^{-1}$ is really the direct function and log the inverse. We have then $\log^{-n}\log^{n} a = a$ for all values, but one value at least can always be found such that $\log^{n}\log^{-n} a = a$.

For the expression $a + a + \ldots$ to b terms, we shall write $a \underset{1}{+} b$; and for $a \underset{1}{+} a \underset{1}{+} a \underset{1}{+} \ldots$ to $\log b$ terms, we shall write $a \underset{2}{+} b$; and in general

$$a \underset{n}{+} a \underset{n}{+} a \underset{n}{+} \ldots \text{ to } \log^{n} b \text{ terms} = a \underset{n+1}{+} b. \tag{5}$$

The (n + 1)*th process applied to two quantities is equal to the inverse log of the* n*th process applied to the logs of the quantities.* That is,

$$a \underset{n+1}{+} b = \log^{-1}(\log a \underset{n}{+} \log b). \tag{6}$$

We shall first show that this proposition is true for $n = 0$ and for $n = 1$. Taking the inverse log of both members of (4), we have, by (5),

$$a \underset{1}{+} b = \log^{-1}(\log a + \log b)$$

Hence

$$\begin{aligned} a \underset{1}{+} a \underset{1}{+} \ldots \text{ to } \log b \text{ terms} &= \log^{-1}(\log a + \log a + \ldots \text{ to } \log b \text{ terms}) \\ = a \underset{2}{+} b \quad &= \log^{-1}(\log a \underset{1}{+} \log b). \end{aligned}$$

But if (6) is true for any value n, viz. if

$$a \underset{n}{+} b = \log^{-1}(\log a \underset{n-1}{+} \log b), \tag{7}$$

then it is also true for $n + 1$. For, by (5),

$$\begin{aligned} a \underset{n+1}{+} b &= a \underset{n}{+} a \underset{n}{+} \ldots \text{ to } \log^{n} b \text{ terms}, \\ &= \log^{-1}[\log a \underset{n-1}{+} \log a \underset{n-1}{+} \ldots \text{ to } \log^{n-1}(\log b) \text{ terms}] \\ &= \log^{-1}[\log a \underset{n}{+} \log b] \end{aligned}$$

by (7) and (5). It has been shown to be true for $n = 0$ and $n = 1$, and hence it is true for every value of n.

By repeated applications of this proposition, we have

$$\begin{aligned} a \underset{n}{+} b &= \log^{-1}(\log a \underset{n-1}{+} \log b), \\ \log a \underset{n-1}{+} \log b &= \log^{-1}(\log^{2} a \underset{n-2}{+} \log^{2} b), \\ &\cdots\cdots\cdots \\ \log^{p-1} a \underset{n-p+1}{+} \log^{p-1} b &= \log^{-1}(\log^{p} a \underset{n-p}{+} \log^{p} b), \end{aligned}$$

whence, by successive substitutions,

$$a \underset{n}{+} b = \log^{-p}(\log^{p} a \underset{n-p}{+} \log^{p} b), \tag{8}$$

and in particular, when $p = n$,

$$a \underset{n}{+} b = \log^{-n}(\log^{n} a + \log^{n} b) \tag{9}$$

or

$$\log^{n}(a \underset{n}{+} b) = \log^{n} a + \log^{n} b, \tag{10}$$

or, in words, *the* n*th process applied to two quantities is equal to the inverse* n*th log of the sum of the* n*th logs of the quantities.* (9) and (6) are the most important properties of the n^{th} process.

Taking the log of both members of (7), we have

$$\log a \underset{n-1}{+} \log b = \log (a \underset{n}{+} b).$$

If we substitute $\log^{-1} a$ for a and $\log^{-1} b$ for b, this becomes

$$\log \log^{-1} a \underset{n-1}{+} \log \log^{-1} b = \log (\log^{-1} a \underset{n}{+} \log^{-1} b),$$

or

$$a \underset{n-1}{+} b = \log (\log^{-1} a \underset{n}{+} \log^{-1} b). \tag{11}$$

So, applying (8) to $\log^{-p} a$ and $\log^{-p} b$, and taking $\log^{p}$ of both members of the equation, we have

$$a \underset{n-p}{+} b = \log^{p} (\log^{-p} a \underset{n}{+} \log^{-p} b), \tag{12}$$

and in particular, when $p = n$,

$$a + b = a \underset{0}{+} b = \log^{n} (\log^{-n} a \underset{n}{+} \log^{-n} b) \tag{13}$$

or

$$\log^{-n} (a \underset{0}{+} b) = \log^{-n} a \underset{n}{+} \log^{-n} b.$$

(11) gives the method of passing from any addition-process to the next lower one, and shows that there is an infinite descending series of such processes. Making $n = 0$ in (12), it becomes

$$a \underset{-p}{+} b = \log^{p} (\log^{-p} a + \log^{-p} b), \tag{14}$$

which gives the general expression for a negative addition process in terms of the common addition process. A negative addition process is of course a very different thing from a subtraction process. $a \underset{-1}{+} b$ bears the same relation to addition that addition bears to multiplication.

There is no essential difference between the + process and the − process. They obey the same laws, and either one may be taken to be the inverse of the

other. Hence the above proofs apply equally well to the $-$ process, and that also presents an infinite descending as well as ascending series. $a \underset{-n+1}{-} b$, for instance, is a command to find the $\log^{-n} b$ equal factors which when united by the $\underset{-n}{+}$ process will give a.

If we write $n + p$ for n, and therefore n for $n - p$ in (8), it becomes, using both signs,

$$a \underset{n+p}{\pm} b = \log^{-p}(\log^{p} a \underset{n}{\pm} \log^{p} b), \tag{15}$$

and this equation contains the whole theory of the positive and negative addition and subtraction processes, or better, of the positive and negative direct and inverse addition processes, provided it be observed that both n and p may be either positive or negative. In fact, (6), (11), (12), (13) are what (15) becomes when $p = 1$, $p = -1$, $p = -p$, $p = -n$, respectively.

In the $\underset{0}{+}$ process, 0 is the symbol that the same quantity has been both added and subtracted, that is, that no effect has been produced. We write $a + b - b = a + 0 = a$, and we may say that 0 is the ineffective term in the $\underset{0}{+}$ process. Making $b = \log^{-n} 0$ in (9), it becomes

$$\begin{aligned} a \underset{n}{+} \log^{-n} 0 &= \log^{-n}[\log^{n} a + \log^{n}(\log^{-n} 0)] \\ &= \log^{-n}(\log^{n} a + 0) = a, \end{aligned}$$

or, *in the* $\underset{n}{+}$ *process, the ineffective term is* $\log^{-n} 0$. It follows that, as by $+ b$ is to be understood $0 + b$, so by $\underset{n}{+} b$ is to be understood $\log^{-n} 0 \underset{n}{+} b$.

For the sake of greater variety and sometimes greater simplicity in means of expression, the ordinary symbols of algebra may now be introduced. In accordance with the definition of the $\underset{1}{+}$ and the $\underset{2}{+}$ process, we have

$$\begin{array}{lll} a \underset{0}{+} b = a + b, & a \underset{1}{+} b = a \times b = ab, & a \underset{2}{+} b = a^{\log b} \\ a \underset{0}{-} b = a - b, & a \underset{1}{-} b = a \div b = \frac{a}{b} & a \underset{2}{-} b = a^{\frac{1}{\log b}}. \end{array}$$

Introduce the symbol base by means of the equation $\log^{-1} a = e^{a}$. Substitute $\log^{-1} a$ for a in this equation, and it becomes

$$\log^{-1}(\log^{-1} a) = \log^{-2} a = e^{\log^{-1} a} = e^{e^{a}},$$

and by repeating the same process

$$\begin{aligned} \log^{-n} a &= e^{e^{\cdot^{\cdot^{\cdot^{e^{a}}}}}} \text{ to } n \text{ exponents } e, \\ &= e^{e^{\cdot^{\cdot^{\cdot^{e^{\log^{-p} a}}}}}} \text{ to } (n - p) \text{ exponents } e, \end{aligned} \tag{16}$$

and taking the inverse p^{th} log of both members of (16) and then writing $-n-p=-m$, and hence $n=m-p$, we have

$$\log^{-m} a = \log^{-p}\left(e^{\cdot^{\cdot^{\cdot^{e^a}}}} \text{ to } (m-p)\ e\text{'s}\right).$$

As particular cases of (16) we have

$$\log^{-n} 0 = e^{e^{\cdot^{\cdot^{\cdot^{e}}}}} \text{ to } (n-1)\ e\text{'s}, \qquad \log^{-n} 1 = e^{\cdot^{\cdot^{\cdot^{e}}}} \text{ to } n\ e\text{'s},$$

$$\log^{-n} e = e^{\cdot^{\cdot^{\cdot^{e}}}} \text{ to } (n+1)\ e\text{'s},$$

$$\log^{-n}\left(e^{\cdot^{\cdot^{\cdot^{e}}}} \text{ to } m\ e\text{'s}\right) = e^{e^{\cdot^{\cdot^{\cdot^{e}}}}} \text{ to } (n+m)\ e\text{'s};$$

and hence, for various expressions for the ineffective term in the $\underset{n}{+}$ process, we have

$$\log^{-n} 0 = \log^{-(n-1)} 1 = \log^{-(n-2)} e = \log^{-(n-3)} e^e = \ldots.$$

The entire body of equivalent expressions will then be

$$a \underset{-n}{\pm} b = \log^n(\log^{-n} a \pm \log^{-n} b) = \log^n\left(e^{e^{\cdot^{\cdot^{\cdot^{a}}}}} \pm e^{e^{\cdot^{\cdot^{\cdot^{b}}}}} \text{ to } n\ e\text{'s}\right)$$

. . . .

$$a \underset{-3}{\pm} b = \log^3(\log^{-3} a \pm \log^{-3} b) = \log^3\left(e^{e^{e^a}} \pm e^{e^{e^b}}\right)$$

$$a \underset{-2}{\pm} b = \log^2(\log^{-2} a \pm \log^{-2} b) = \log^2\left(e^{e^a} \pm e^{e^b}\right)$$

$$a \underset{-1}{\pm} b = \log(\log^{-1} a \pm \log^{-1} b) = \log(e^a \pm e^b)$$

$$a \underset{0}{\pm} b = \log^0(\log^0 a \pm \log^0 b) = a \pm b \tag{17}$$

$$a \underset{1}{\pm} b = \log^{-1}(\log a \pm \log b) = e^{\log a \pm \log b} = a \overset{\times}{\div} b = ab \text{ or } \frac{a}{b}.$$

$$a \underset{2}{\pm} b = \log^{-2}(\log^2 a \pm \log^2 b) = e^{e^{\log^2 a \pm \log^2 b}} = a^{\log b} \text{ or } a^{\frac{1}{\log b}} = e^{\log a \overset{\times}{\div} \log b}$$

. . . .

$$a \underset{n}{\pm} b = \log^{-n}(\log^n a \pm \log^n b = e^{e^{\cdot^{\cdot^{\cdot^{e^{\log^n a \pm \log^n b}}}}}} \text{ to } n\ e\text{'s} = e^{e^{\cdot^{\cdot^{\cdot^{e^{\log^{n-m} a \underset{m}{\pm} \log^{n-m} b}}}}}} \text{ to } (n-m)\ e\text{'s}.$$

The $\underset{n}{+}$ process can be expressed in one other way in terms of ordinary algebra. We have

$$b^{\log b^{\log^2 b^{\log^3 b^{\log^4 a - 1} - 1} - 1}} = \log^{-1}\left[\log b^{\log^2 b^{\log^3 b^{\log^4 a - 1} - 1} - 1} \times \log b\right]$$

$$= \log^{-2}\left[\left(\log^2 b^{\log^3 b^{\log^4 a - 1} - 1} - 1\right) \log^2 b + \log^2 b\right]$$

$$= \log^{-2}\left(\log^2 b^{\log^3 b^{\log^4 a - 1} - 1} \times \log^2 b\right) = \log^{-3}\left[\left(\log^3 b^{\log^4 a - 1} - 1\right) \log^3 b + \log^3 b\right]$$

$$= \log^{-4}\left[(\log^4 a - 1) \log^4 b + \log^4 b\right] = \log^{-5}(\log^5 a + \log^5 b)$$

$$= a \underset{5}{+} b.$$

By substituting $\log a$, $\log b$ for a, b in the above expression, and taking the $\log^{-1}$ of the result, the value of $a \underset{6}{+} b$ is obtained, and hence the value of $a \underset{n}{+} b$. In the same way, it may be shown that for the inverse n^{th} process we have, if we write $\div \log b$ for $1 \div \log b$,

$$a \underset{n}{-} b = a^{\log a^{\log^2 a^{\cdot^{\cdot^{\cdot^{\div \log^{n-1} b - 1}}}} - 1}}$$

The $\underset{0}{+}$ process has been defined to be a process subject to the principles of commutation and distribution expressed by

$$(a + b) + c = a + (c + b). \qquad (3)$$

But all the $\underset{n}{+}$ processes are subject to the same conditions, for

$$\begin{aligned}(a \underset{n}{+} b) \underset{n}{+} c &= \log^{-n}[\log^n(a \underset{n}{+} b) + \log^n c] \\ &= \log^{-n}[\log^n a + \log^n b + \log^n c] \\ &= \log^{-n}[\log^n a + (\log^n c + \log^n b)] \\ &= \log^{-n}[\log^n a + \log^n(c \underset{n}{+} b)] \\ &= a \underset{n}{+} (c \underset{n}{+} b) \end{aligned} \tag{18}$$

When $n = 1$, this becomes

$$abcde = (ab)\,c\,(de) = (abcd)\,e = (abe)\,(cd),$$

and it gives the simplest possible proof of a proposition to the demonstration of which four pages are devoted in Lejeune-Dirichlet's *Zahlentheorie*, edited by Dedekind. It depends only upon the fact that if

$$(a + b + c + d + e) = (a + b) + c + (d + e) = (a + b + e) + (c + d)$$

is true for all values of $a, b \ldots.$, then it is true when for $a, b \ldots.$ we substitute $\log a$, $\log b \ldots.$

When we reach evolution, we are in the habit of supposing that the principle of commutation no longer applies, but that is simply the fault of the unsymmetrical character of our notation. It is true that a^b is not equal to b^a, but a^b ought not to be regarded as explicitly a function of a and b, but of a and $\log^{-1} b$, and it is a function of these quantities such that a and $\log^{-1} b$ are interchangeable. It is $a^{\log(\log^{-1} b)}$ and that is equal to $(\log^{-1} b^{\log a})$. In applying the principle to mixed direct and inverse processes, it is necessary to take account of suppressed signs. $a - b$, which is in full $+ a - b$, is, of course, not equal to $+ b - a$, but to $- b + a$. So $a^{\frac{1}{b}}$ becomes, on changing the order of the terms, $(e^{\frac{1}{b}})^{\log a}$.

Since it is now seen that the distinguishing mark of the $\underset{0}{+}$ process applies equally well to every process, it follows that the process of ordinary addition has not that fundamental character which we are accustomed to attribute to it. The whole theory of algebra is contained in (18), which defines any process by itself, and (5), which gives the connecting link between any process and the next higher or the next lower one. Three successive processes taken anywhere in the infinite series would serve to contain all the knowledge which has actually been expressed in terms of addition, multiplication and involution.

So far I have expressed the general process in terms of addition. It might have been expressed equally well in terms of multiplication or involution. The fact that the multiplication idea has already received greater extension than either of the other two (since $f(a)$ is properly regarded as the product of the

quantity a by the function f) might afford a reason for preferring that as the basis of the generalized idea, were it not that the existing expression for addition and its inverse, in both language and symbols, is superior to that for multiplication and its inverse. $a \pm b$ is easier to write and easier to read than $a \overset{\times}{\div} b$ or $a \divideontimes b$. The signs $+$ and $-$ are themselves defective. There is great advantage in having every symbol of such a form that its opposite can be indicated by actually turning it over, as Mr. Peirce's symbol for the copula, which is such that $A \prec B$ means *A is wholly contained in B*, and $A \succ B$ means *A wholly contains B*. It is an advantage possessed by many of the letters of the alphabet, but not yet made use of. For instance, a substitution and its inverse might be denoted by ψ and $\rotatebox{180}{$\psi$}$, and any function and its inverse by φ and $\rotatebox{180}{$\varphi$}$. Direct and inverse operations have, in fact, the same relation to each other as the world inside and the world outside a mirror; which is to be taken as direct and which as inverse depends only on the point of view. If $+$ had had a slightly different form, say $\top$, then $-$ might have been written $\perp$, but at the time when the sign $+$ was adopted, subtraction, far from being considered as something on a par with addition, was called the *vitium negationis*, and was avoided in all possible ways. Until, then, mathematical words and signs have been made over again on scientific principles, instead of being perpetuated as they have sprung into existence by chance, $\underset{s}{\pm}$ is the best form of expression for the general process.

But it is to be observed that everything can be expressed in terms of $\times$ and $\div$ at any moment, *and without changing the index*, provided we agree to call ordinary addition the 0-addition and ordinary multiplication the 1-multiplication (and ordinary involution the 2-involution). The successive processes would be $\underset{0}{+}$, $\underset{1}{\times}$, and $(\ )_2^{\log(\ )}$, and we should have *

$$\underset{n}{+} = \underset{n}{\times} = (\ )_n^{\log(\ )}.$$

It is to be observed that the inverse of $a + b$ regarded as a function of a is not the same as its inverse regarded as a function of b. In general, the inverse

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* I am not aware that any one but De Morgan has considered the extension, both forwards and backwards, of the three processes which algebra takes account of. De Morgan approaches the subject from a different point of view from that which I have here adopted. He looks at $a + b$ and $a \times b$, and proceeds to construct new processes from the "hints" which they convey. This is doubtless the way in which new processes have usually been arrived at in the first instance, but when they have once been obtained, the most natural thing to do is to look at old and new together, abstract that part of them which is common to both, and use that as their common foundation. Formula (5), which expresses one of the two properties which each of these processes has in common with every other, has not been noticed by De Morgan at all.

De Morgan gives formulæ (6), (9), (11), (14), (16), and (18), in a notation which differs only slightly from mine. He writes indifferently $A \,{}^{(\mathrm{IV})}\, B$, $A +_{\mathrm{IV}} B$, and $A \times_{\mathrm{III}} B$, for what I have called $A \underset{3}{+} B$, and he uses the "scalar function" and its inverse, λ and γ, for what I have at once identified with the logarithm and its inverse.

of an expression regarded as a function of any quantity p which enters it (which we shall denote by I_p) is f^{-1} in $f^{-1}f(p,r) = p$, while its inverse with respect to r is f^{-1} in $f^{-1}f(p,r) = r$. For example,

$$I_a(a+b) = a-b, \qquad I_b(a+b) = -a+b.$$

We have $I_a(a^b) = a^{\frac{1}{b}}$. Let us proceed to find $I_b(a^b)$; that is, a function of b such that when applied to a^b instead of b it will give b. We have

$$I_b(a^b) = I_b\left[(\log^{-1} b)^{\log a}\right].$$

This is a command to take the inverse log of b and to raise that to the power $\log a$. To invert the process (since if $f = \phi\psi$, then $f^{-1} = \psi^{-1}\phi^{-1}$ and not $\phi^{-1}\psi^{-1}$) we must raise b to the power $1 \div \log a$ and take the direct log of the result; that is,

$$I_b(a^b) = \log\left(b^{\frac{1}{\log a}}\right) = \frac{\log b}{\log a} = e^{\log^2 b - \log^2 a},$$

and in particular, when we make $a = e$, we have

$$I_b(e^b) = e^{\log^2 b}.$$

So, more generally,

$$I_a\left[f(a) \underset{n}{+} \phi(b)\right] = f^{-1}\left[a \underset{n}{+} \phi^{(b)}\right]. \tag{19}$$

If in $e^{\log^2 a - \log^2 b}$ we substitute a^b for b, we recover b, as we should.

To take another example, since

$$I_c(a^b)^c = \log c \div b \log a,$$

we have

$$\begin{aligned} I_D(e^{h\cdot D}) &= I_D f(x+h) = \log D \div h \\ &= \frac{1}{h}\left[(D-1) - \frac{1}{2}(D-1)^2 + \cdots\right] \\ &= \frac{1}{h}\left[\log\frac{E}{e} - \frac{1}{2}\left(\log\frac{E}{e}\right)^2 + \cdots\right] \end{aligned}$$

since

$$D - 1 = \log E - \log e = \log\frac{E}{e}\cdot$$

De Morgan remarks (Trigonometry and Double Algebra, p. 166) that one operation and its inverse and the scalar function and its inverse are sufficient for expression; and he then proceeds to base the entire series of operations upon the equation of condition

$$(a \underset{n}{+} b) \underset{n+1}{+} c = (a \underset{n+1}{+} c) \underset{n}{+} (b \underset{n+1}{+} c), \tag{20}$$

which is the generalization of $(a+b)c = ac + bc$. But this is not of the nature of a definition of the $\underset{n+1}{+}$ process. It is a rule for the distribution of a term affected by the sign $\underset{n+1}{+}$ over any number of terms affected by the sign $\underset{n}{+}$, and it is of no more fundamental importance than that for the distribution of a term affected by $\underset{n+2}{+}$ over terms affected by $\underset{n}{+}$. Both are easy consequences of the definition (5). $(a+b)c = ac + bc$ does not constitute a definition of the multiplication-process; it expresses one of its properties, but not one from which all others can be deduced.

To prove (20), we have, by (5),

$$\begin{aligned}(a \underset{n}{+} b) \underset{n+1}{+} c &= (a \underset{n}{+} b) \underset{n}{+} (a \underset{n}{+} b) \underset{n}{+} \cdots\cdot \text{ to } \log^n c \text{ terms,}\\ &= (a \underset{n}{+} a \underset{n}{+} a \cdots\cdot \text{ to } \log^n c \text{ terms}) \underset{n}{+} (b \underset{n}{+} b \underset{n}{+} \cdots\cdot \text{ to } \log^n c \text{ terms}),\\ &= (a \underset{n+1}{+} c) \underset{n}{+} (b \underset{n+1}{+} c).\end{aligned}$$

When $n = -1, 0, +1, +2$, this gives respectively

$$\log(e^a + e^b) + c = \log(e^{a+c} + e^{b+c}),$$

$$(a+b)c = ac + bc,$$

$$(ab)^c = a^c \,.\, b^c, \qquad \text{or} \qquad c^{a+b} = c^a \,.\, c^b, \tag{21}$$

$$\log^3 c + \log(\log^2 a + \log^2 b) = \log(\log^2 c \log^2 a + \log^- c \log^2 b).$$

The two equations (21), although they seem to express very different facts, do really differ from each other just as much as, and no more than, $c(a+b)$ and $(a+b)c$.

There is no method for the distribution of a term affected by any sign over terms affected by the next higher sign; for instance, $ab + c$ cannot be expressed as $\phi[f(a,c), f(b,c)]$.

De Morgan states in effect* that all the theorems of ordinary algebra which can be expressed in terms of addition and multiplication hold for all two successive processes, n and $n+1$, if numerical coefficients are replaced by their n^{th} inverse logs. The complete proposition is this: All the theorems of ordinary algebra (which are theorems in addition, multiplication, and evolution) hold for the processes n, $n+1$, $n+2$, provided numerical summands and factors are replaced by their n^{th} inverse logs, and numerical exponents by their $(n+1)^{\text{th}}$ inverse logs. For, given an equation expressing a theorem of ordinary algebra; it continues to be true when for the literal quantities which enter it, their n^{th} logs are substituted, and it still continues to be true when the $\log^{-n}$ of each

* Trigonometry and Double Algebra, p. 166.

member is taken. But the effect of these two operations is to substitute for $+$, $\times$, and $(\ )^{(\ )}$ the processes n, $n+1$, $n+2$, on the literal terms, the $\log^{-n}$ of numbers not exponents, and the $\log^{-(n+1)}$ of exponents. Say $f(b) = 5 + 4\,(b^3)$, then

$$
\begin{aligned}
f_n(b) &= \log^{-n}\,[5 + 4\,(\log^n b)^3] \\
&= \log^{-n}\,[\log^n \log^{-n} 5 + \log^n \log^{-n}\,(4\,\overline{\log^n b}^{\,3})]\,; \qquad (22)
\end{aligned}
$$

but

$$
\begin{aligned}
\log^{-n}\,(4\,\overline{\log^n b}^{\,3}) &= \log^{-(n+1)}\,[\log 4 + \log \overline{\log^n b}^{\,3}] \\
&= \log^{-(n+1)}\,[\log^{n+1} \log^{-n} 4 + \log^{n+1} \log^{-n} \overline{\log^n b}^{\,3}], \qquad (23)
\end{aligned}
$$

and

$$
\begin{aligned}
\log^{-n}\,(\overline{\log^n b}^{\,3}) &= \log^{-(n+1)}\,[3 \log^{n+1} b] \\
&= \log^{-(n+2)}\,[\log 3 + \log^{n+2} b] \\
&= \log^{-(n+2)}\,[\log^{n+2} \log^{-(n+1)} 3 + \log^{n+2} b] \\
&= \log^{-(n+1)} 3 \underset{n+2}{+} b,
\end{aligned}
$$

which in (23) gives

$$
\begin{aligned}
\log^{-n}\,(4\,\overline{\log^n b}^{\,3}) &= \log^{-(n+1)}\,[\log^{n+1} \log^{-n} 4 + \log^{n+1}(\log^{-(n+1)} 3 \underset{n+2}{+} b)] \\
&= \log^{-n} 4 \underset{n+1}{+} (\log^{-(n+1)} 3 \underset{n+2}{+} b)
\end{aligned}
$$

which in (22) gives

$$
\begin{aligned}
\log^{-n}\,[5 + 4\,(\log^n b)^3 &= \log^{-n}\{\log^n \log^{-n} 5 + \log^n\,[\log^{-n} 4 \underset{n+1}{+} (\log^{-n+1} 3 \underset{n+2}{+} b)]\} \\
&= \log^{-n} 5 \underset{n}{+} [\log^{-n} 4 \underset{n+1}{+} (\log^{-(n+1)} 3 \underset{n+2}{+} b)], \qquad (24)
\end{aligned}
$$

which was to be proved. But nothing in this proof depends on the particular summand, factor, and exponent, 5, 4, and 3; hence the proposition is true in general. The reason that numbers which enter the original function as exponents are replaced by their $\log^{-n} \log^{-1}$ instead of by their $\log^{-n}$ is that the exponential function is in reality a function of the $\log^{-1}$ of the exponent, as has appeared before.

EXAMPLE. $(a + b - c)\,(a - b + c) = a^2 - b^2 - c^2 + 2\,bc$ gives in the next stage

$$
\left(\frac{ab}{c}\right)^{\log\left(\frac{ac}{b}\right)} = a^{\log a} \div b^{\log b} \div c^{\log c} \times 2^{\log b \log c}
$$

and in general

$$
(a \underset{n}{+} b \underset{n}{-} c) \underset{n+1}{+} (a \underset{n}{-} b \underset{n}{+} c) = (\underset{n}{2} \underset{n+1}{+} b \underset{n+1}{+} c) \underset{n}{+} (a \underset{n+2}{+} \underset{n+1}{2}) \underset{n}{-} (b \underset{n+2}{+} \underset{n+1}{2}) \underset{n}{-} (c \underset{n+2}{+} \underset{n+1}{2})
$$

if we write $\underset{n}{2}$ for $\log^{-n} 2$ and $\overset{n}{2}$ for $\log^n 2$.

It is not necessary to write down the generalization of ordinary algebraical theorems. As every equation in analytical geometry is susceptible of a double interpretation according as its variables are looked at as trilinear or tangential co-ordinates, so every algebraical equation may be regarded as the receptacle of an infinite number of propositions concerning the infinite number of possible three adjacent processes.

The generalization of the binomial theorem is the law of distribution of c over a and b in $(a \underset{n}{+} b) \underset{n+2}{+} c$. We shall write $\sum_{n}^{+}$ to indicate the summation of a number of terms connected by the sign $\underset{n}{+}$; then $\sum_{1}^{+}$ denotes a product, $\sum_{2}^{+}$ a series of powers, etc. The binomial theorem is

$$(a+b)^p = \sum_{0\ r=p}^{+\ r=0} \left[\frac{p\,.\,(p-1)\,\ldots\,.\,(p-\overline{r-1})}{1\,.\,2\,\ldots\,.\,r}\, a^{p-r} b^r,\right],$$

and, applying the rule for generalization, we obtain

$$(a \underset{n}{+} b) \underset{n+2}{+} c = (a \underset{n}{+} b) \underset{n+1}{+} (a \underset{n}{+} b) \underset{n+1}{+} \ldots . \text{ to } (\log^{n+1} c\,, = p\,,) \text{ terms} \quad (25)$$

$$= \sum_{n\ r=p}^{+\ r=0} [p \underset{n+1}{+} (p \underset{n}{-} \log^{-n} 1) \underset{n+1}{+} \ldots (p \underset{n}{-} \log^{-n} \overline{r-1}) \underset{n+1}{-} \log^{-n} 1 \underset{n+1}{-} \log^{-n} 2 \ldots \underset{n+1}{-} \log^{-n} r$$
$$\underset{n+1}{+} (a \underset{n+2}{+} \log^{-n-1} \overline{p-r}) \underset{n+1}{+} (b \underset{n+2}{+} \log^{-n-1} r)] \quad (26)$$

in which p is to be replaced by $\log^{n+1} c$. The immediate proof of this theorem is just as easy as that of the binomial theorem. In fact, it is exactly the same, since it consists in the application of (20) to the second number of (25). This may then be established first, and the binomial theorem taken as a special case of it.

The generalization of

$$(a+b)^3 = a^3 + 3\,a^2 b + 3\,ab^2 + b^3$$

is, if we write, with De Morgan, $\underset{n}{3}$ for $\log^{-n} 3$,

$$a \underset{n}{+} b \underset{n+2}{+} \underset{n+1}{3} = a \underset{n+2}{+} \underset{n+1}{3} \underset{n}{+} b \underset{n+2}{+} \underset{n+1}{3} \underset{n}{+} \underset{n}{3} \underset{n+1}{+} b \underset{n+1}{+} a \underset{n+2}{+} \underset{n+1}{2} \underset{n}{+} \underset{n}{3} \underset{n+1}{+} a \underset{n+1}{+} b \underset{n+2}{+} \underset{n+1}{2}.$$

For $n = 1$, this gives

$$(ab)^{\overline{\log ab^2}} = a^{\overline{\log a^2}} \times b^{\overline{\log b^2}} \times (b^3)^{\overline{\log a^2}} \times (a^3)^{\overline{\log b^2}},$$
$$= (ab^3)^{\overline{\log a^2}} (ba^3)^{\overline{\log b^2}},$$

and, with the negative sign,

$$\left(\frac{a}{b}\right)^{\left(\log \frac{a}{b}\right)} = (a \div b^3)^{\overline{\log a^2}} \div (b \div a^3)^{\overline{\log b^2}}.$$

De Morgan says* that in the farther consideration of the higher processes new inexplicables might, and perhaps would, arise. They do in fact arise, and in the following manner. It is necessary first to establish the lemma

$$\log^p(\underset{n}{\pm}\, a) = \underset{n-p}{\pm} \log^p a. \tag{27}$$

If we make b equal to the inefficient term of the n^{th} process in

$$b \underset{n}{\pm} a = \log^{-p}(\log^p b \underset{n-p}{\pm} \log^p a),$$

we have

$$\log^{-n} 0 \underset{n}{\pm} a = \underset{n}{\pm}\, a = \log^{-p}(\log^p \log^{-n} 0 \underset{n-p}{\pm} \log^p a);$$

but $\log^{p-n} 0$ is the inefficient term of the $(n-p)^{\text{th}}$ process, hence, taking $\log^p$ of both members,

$$\log^p(\underset{n}{\pm}\, a) = \underset{n-p}{\pm} \log^p a$$

and

$$\log^n(\underset{n}{\pm}\, a) = \pm \log^n a.$$

In particular,

$$\log(\overset{\times}{\div}\, a) = \pm \log a, \quad \text{and} \quad \log^2[(\ )^a \text{ or } (\ )^{\frac{1}{a}}] = \pm \log^2 a.$$

We have, therefore,

$$\begin{aligned}
(\underset{n}{-}\, a) \underset{n+1}{+} (\underset{n}{-}\, a) &= \log^{-n}[\log^n(\underset{n}{-}\, a) \underset{1}{+} \log^n(\underset{n}{-}\, a)] \\
&= \log^{-n}[(-\log^n a) \times (-\log^n a)] \\
&= \log^{-n}[(+\log^n a) \times (+\log^n a)] \\
&= \log^{-n}[+\log^n a \,.\, \log^n a] \\
&= (\underset{n}{+}\, a) \underset{n+1}{+} (\underset{n}{+}\, a) = \underset{n}{+}(a \underset{n+1}{+} a) = \underset{n}{+}\, p.
\end{aligned}$$

In the same way it may be shown that

$$(\underset{n}{+}\, a) \underset{n+1}{+} (\underset{n}{-}\, a) = \underset{n}{-}(a \underset{n+1}{+} a) = \underset{n}{-}\, p. \tag{28}$$

It appears, then, that $\underset{n}{+}(a \underset{n+1}{+} a)$ is made up of two equal terms connected by the sign $\underset{n+1}{+}$, being either both $(\underset{n}{+}\, a)$ or both $(\underset{n}{-}\, a)$, but $\underset{n}{-}(a \underset{n+1}{+} a)$ cannot possibly be made up of two equal terms connected by the sign $\underset{n+1}{+}$. If, then, a solution be demanded of the system of equations

$$x \underset{n+1}{+} y = \underset{n}{-}\, p, \qquad\qquad x = y,$$

the answer is that the value of the variable is imaginary, and an imaginary of the n^{th} kind. In particular, the value of x in

* Camb. Phil. Trans. VIII. 142.

$$x \,.\, x = -p, \qquad x^{\log x} = 1 \div p, \qquad x^{\log x^{\log^2 x - 1}} = e^{\div \log p}$$

is an imaginary of the first, second, and third kinds respectively. But while these expressions are, in the strict sense of the term, imaginaries of a higher kind, it is none the less possible to express them as functions of $\sqrt{-1}$. The solutions of

$$x \,.\, x = 0 - 1, \qquad x^{\log x} = 1 \div c, \qquad x^{\log x^{\log^2 x - 1}} = e^{\div e}$$

are respectively

$$x = i, \qquad x = e^i, \qquad x = e^{e^i}$$

and in general

$$x \underset{n+1}{+} x = \underset{n}{-} \log^{-(n+1)} 0$$

gives

$$x = \log^{-n}, \qquad i = e^{e^{\cdot^{\cdot^{\cdot^{e^i}}}}} \text{ to } n\ e\text{'s.}$$

or the $(n+1)^{\text{th}}$ imaginary is $\log^{-n} i$.

The fact that the high imaginaries are inverse logarithms of the first imaginary corresponds to the fact that the high processes are themselves nothing more than high inverse logarithms of the sums of high logarithms.

The entire body of algebra may now be taken as proved for any process, and for any succession of processes in the infinite series. We proceed to consider differentiation. It should first be shown that, u, v, and w being functions of x,

$$D_x(u + v + w) = D_x u + D_x v + D_x w.$$

The derivative of log should next be obtained by expanding $\log(x + \Delta x) - \log x$ in series and passing to the limit. From that the derivatives of all algebraic functions easily follow. We have

$$D_x \log x = \frac{1}{x}, \qquad D_x \log^2 x = D_x \log(\log x) = \frac{D_x \log x}{\log x} = \frac{1}{x \log x},$$

$$D_x \log^n x = \frac{D_x \log^{n-1} x}{\log^{n-1} x} = \frac{D_x \log^{n-2} x}{\log^{n-2} x \,.\, \log^{n-1} x} = \dots\dots = \frac{1}{x \,.\, \log x \,.\, \log^2 x \dots\dots \log^{n-1} x}.$$

Putting $\log^{-1} z = y$, we have $z = \log y$, $D_x z = D_x \log y = \frac{D_x y}{y}$, whence

$$D_x y = D_x \log^{-1} z = \log^{-1} x \,.\, D_x z$$

and

$$D_x \log^{-1}(\log^{-1} z) = \log^{-1}(\log^{-1} z)\, D_x \log^{-1} z = \log^{-2} z \log^{-1} z\, D_x z$$

and

$$D_x \log^{-n} x = \log^{-n} x \log^{-n+1} x \dots\dots \log^{-1} x.$$

By this means we obtain

$$\begin{aligned}
&D_x(u \underset{n}{+} v) = D_x \log^{-n}(\log^n u + \log^n v) \\
&= \log^{-n}(\log^n u + \log^n v) \,.\, \log^{-n+1}(\log^n u + \log^n v) \ldots\ldots \log^{-1}(\log^n u + \log^n v) \text{ into} \\
&= (u \underset{n}{+} v)(\log u \underset{n-1}{+} \log v) \ldots\ldots (\log^{n-1} u \underset{1}{+} \log^{n-1} v) \text{ into} \\
&= (u \underset{n}{+} v) \log^{-1}(u \underset{n}{+} v) \log^{-2}(u \underset{n}{+} v) \ldots\ldots \log^{-n+1}(u \underset{n}{+} v) \text{ into}
\end{aligned}$$

where the common multiplier is

$$\left(\frac{D_x u}{u \log u \log^2 u \ldots\ldots \log^{n-1} u} + \frac{D_x v}{v \log v \ldots\ldots \log^{n-1} v}\right).$$

When $n = 1$,

$$D_x(u \underset{1}{+} v) = D_x(uv) = (u \underset{1}{+} v)\left(\frac{D_x u}{u} + \frac{D_x v}{v}\right) = uD_x v + vD_x u.$$

When $n = 2$,

$$\begin{aligned}
D_x(u \underset{2}{+} v) = D_x(u^{\log v}) = D_x(v^{\log u}) &= u^{\log v}(\log u \,.\, \log v)\left(\frac{D_x u}{u \log u} + \frac{D_x v}{v \log v}\right) \\
&= \log v \,.\, u^{\log v - 1} D_x u + \log u \,.\, v^{\log u - 1} D_x v.
\end{aligned}$$

Substituting in this z for $\log v$, and hence $\log^{-1} z D_x z$ for $D_x v$, this becomes

$$\begin{aligned}
D_x(u^z) &= z \,.\, u^{z-1} D_x u + \log u \,.\, \log^{-1} z^{\log u} D_x z \\
&= z \,.\, u^{z-1} D_x u + \log u \,.\, u^z D_x z.
\end{aligned}$$

When we make $z = n$, $u = a$, $u = z = x$, $u = e$, we have respectively

$$D_x(u^n) = nu^{n-1} D_x u, \qquad D_x(a^z) = \log a \,.\, a^z D_x z$$
$$D_x(x^x) = x^x(1 + \log x).$$
$$D_x(e^z) = e^z D_x z.$$

Having once obtained the value of $D_x(u + v)$ and of $D_x \log x$, we are able to avoid the necessity of ever again passing to the limit.

There is no real reason for restricting algebra to the consideration of three successive processes. The next advance should be to take in the relations of four processes, and in particular to obtain in a simple form the rule for the distribution of c in

$$\begin{aligned}
(a \underset{0}{+} b) \underset{3}{+} c, &= e^{\log(a+b)^{\log^2 c}} \\
&= c^{\log c^{\log^2(a+b)}} \div c^{\log c}.
\end{aligned}$$

On the Motion of a Perfect Incompressible Fluid when no Solid Bodies are Present.

BY HENRY A. ROWLAND,
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CONTENTS.

IN a paper published in this Journal recently, the analogy of an electric current in fluid motion was found to be what I there called the *vortices* of the *vortices* of the fluid. It soon became evident to me that the kind of motion there referred to was a motion which was higher than vortex motion and yet had some of its properties.

The first portion of the present paper, treating of vector quantities in general, shows how to deduce the higher vectors from the lower, and also gives a method by which from three integrals any lower system can be deduced from any higher system. One of the most useful theorems, both for use in hydrodynamics and electro-magnetism, is that which shows how to replace any system of vectors by another system having the same external effect or the same internal effect, as the case may be. Another useful theorem shows how to deduce a system of vectors satisfying the equation of continuity from one which does not, though the principle of this theorem has long been recognized in the idea of cyclic constants. But I believe it will be found more useful as stated in this paper.

Application is then made to the kinematics of fluid motion.

Fluid motion must satisfy the equation of continuity, and so must be always cyclic. Otherwise there must be points somewhere in space where the fluid is created or destroyed.

As Laplace's equation is a form of the equation of continuity, it is evident that this equation cannot, *from a physical point of view*, be satisfied at every point

of space, although there are some functions, like the spherical harmonics of positive degree, that do so mathematically; for these all give some fluid motion, often infinite in amount, at an infinite distance. So that such fluid motion can only be produced by the creation and destruction of fluid in regions infinitely distant. And so all fluid motion in an indefinite medium, without solid bodies, must be due, as it were, to the presence of vortex filaments or filaments of any of the higher motions.

The n^{th} order of motion always satisfies the equation of continuity, and hence the lines of motion are closed. All the infinite series of motions must exist *somewhere* in space, but they may be confined to a point, a line, or a surface, and so, for all other points of space, only the n^{th} and lower orders may exist.

In this case, where the $(n+1)^{th}$ order is zero throughout a certain region, the n^{th} order will have a potential satisfying Laplace's equation throughout that region, though it may be cyclic. The motions of a fluid can be divided into two varieties, rotatory and translatory, the odd orders being rotatory and the even ones translatory.

The motion of the zero order is ordinary irrotational fluid motion; the first order is vortex motion; the second order is what I have called the *vortices* of the *vortices*, or *relative fluid motion*, — in an element possessing this motion the centre moves faster than the sides; the third order is one vortex filament inside another and rotating in the opposite direction: and so we can proceed. We thus get the idea that all the higher motions are, as it were, motions which take place inside the element as it drifts along.

From this we see that when we deal with the higher motions, especially with any discontinuity in their distribution, we must use a smaller element than we otherwise should have to do.

Thus, when we treat a single vortex filament, it is impossible to get the motion of the filament or the energy of the fluid without an integration across the section of the filament, however small this may be. This principle I consider of the utmost importance, and by recognition of it all that has hitherto puzzled us with respect to the motion of a vortex ring or the stability of fluid motion becomes a mere question of ordinary calculation. For, if we attempt to obtain the motion of translation of a circular vortex filament, we only obtain indeterminate results if we treat the filament as a whole. For I here show that the motion of the fluid exterior to the filament is the same for an infinite number of ways in which the vortex motion is distributed within the filament, and each of these ways has its own motion of translation and its own energy.

A method is here given by which any distribution of motion of the n^{th} order can be replaced by another distribution over a surface, so that the action on the

other side of the surface away from the original distribution shall be the same as before. This leads us, in the case of vortex motion, to the formation of hollow vortex rings, or solid ones of any finite cross section, and it is shown that one form of hollow vortex ring is permanent and probably stable.

For I do not admit the validity of Thompson's proof that *all* discontinuous fluid motion is unstable. Illustrations are given of the distributions of motions of the different orders, and it is shown that any such series can represent a number of cases of fluid motion, seeing that we can take any one of the series of motions as the ordinary or irrotational motion.

In treating of the action of forces on a fluid, we see that a single attracting particle can only produce fluid pressure, but no motion. Hence in an unlimited fluid forces having an acyclic potential can have no effect on the fluid motion, but can only produce a distribution of pressure.

When a solid body is immersed in a fluid, the fluid is no longer continuous, but the surface acts like a diaphragm; in this case we can have an acyclic velocity potential outside the body, but cyclic if the body was removed. But where the fluid is unlimited this cannot be so.

The motion of a solid body in a fluid can always be represented by the proper case of motion in an unlimited medium, the condition being that the components of the fluid motion throughout a certain surface representing the surface of the body shall be the same as the components of the motion of an element of the surface of the body at that point, supposing the body to exist.

As the classification of vectors is general, I here apply the method to forces, and so get the idea of forces of different orders.

All forces which can produce fluid motion in an unlimited medium must be cyclic, and must contain all the infinite series of forces somewhere in space, though any given region may contain only forces of the n^{th} and lower orders, the n^{th} having a potential.

Expressions are then given for the fluid pressure in terms of the motion of the fluid, one of which has already been given by Dr. Craig.

I have not yet given any expressions for the energy of the fluid in terms of the higher orders of motion, though that in terms of vortex motion is well known.

An immense number of applications of the ideas here set forth of course suggest themselves, but most of the results so far obtained are set forth in this paper, as I have not yet had time to descend to details. It is of course possible that some of the conclusions will not stand the test of time, but I hope that enough will remain to make the paper of value in all departments of physics where vector quantities are investigated.

The different portions of the paper have not been written in the order here given, and so the same idea is sometimes repeated in different parts.

Neither is the notation very satisfactory, as different letters are sometimes used in the same sense, though in this case the connection is stated. I have changed the notation so often that I despair of perfection, and so publish the paper as it stands.

General Theory of Vector Quantities.

Let there be *any* distribution of a vector quantity, $\bar{M}_0$, throughout space, and let its components be $\bar{F}_0$, $\bar{G}_0$, and $\bar{H}_0$. The dash over the letter signifies that these quantities do not satisfy the equation of continuity. Let us now subject these quantities to the following operations:—

$$F_1 = \frac{d\bar{H}_0}{dy} - \frac{d\bar{G}_0}{dz},$$

$$G_1 = \frac{d\bar{F}_0}{dz} - \frac{d\bar{H}_0}{dx},$$

$$H_1 = \frac{d\bar{G}_0}{dx} - \frac{d\bar{F}_0}{dy},$$

$$F_2 = \frac{dH_1}{dy} - \frac{dG_1}{dz},$$

$$G_2 = \frac{dF_1}{dz} - \frac{dH_1}{dx},$$

$$H_2 = \frac{dG_1}{dx} - \frac{dF_1}{dy},$$

$$F_3 = \frac{dH_2}{dy} - \frac{dG_2}{dz},$$

etc.

And let us continue substitutions of this nature indefinitely.

We have thus derived from $\bar{F}_0$, $\bar{G}_0$, and $\bar{H}_0$ an infinite number of new systems of vectors whose properties and relations to the original vectors are to be studied. In the first place, we notice that all these new quantities satisfy the equation of continuity except $\bar{F}_0$, $\bar{G}_0$, and $\bar{H}_0$, and hence we have

$$\frac{dF_n}{dx} + \frac{dG_n}{dy} + \frac{dH_n}{dz} = 0.$$

From these equations we may find, writing J_0 for the quantity

$$\frac{d\bar{F}_0}{dx} + \frac{d\bar{G}_0}{dy} + \frac{d\bar{H}_0}{dz},$$

the following relations: —

$$F_2 = \frac{dJ_0}{dx} - \Delta^2 \bar{F}_0,$$

$$G_2 = \frac{dJ_0}{dy} - \Delta^2 \bar{G}_0,$$

$$H_2 = \frac{dJ_0}{dz} - \Delta^2 \bar{H}_0,$$

and for the rest,

$$F_n = - \Delta^2 F_{n-2},$$

$$G_n = - \Delta^2 G_{n-2},$$

$$H_n = - \Delta^2 H_{n-2}.$$

Whence, writing χ_0 for the quantity

$$\chi_0 = \frac{1}{4\pi} \iiint \frac{1}{r} \left(\frac{d\bar{F}_0}{dx} + \frac{d\bar{G}_0}{dy} + \frac{d\bar{H}_0}{dz} \right) dxdydz,$$

we have, by Poisson's equation,

$$\bar{F}_0 = \frac{1}{4\pi} \iiint \frac{F_2}{r} \, dx'dy'dz' - \frac{dX_0}{dx},$$

$$\bar{G}_0 = \frac{1}{4\pi} \iiint \frac{G_2}{r} \, dx'dy'dz' - \frac{dX_0}{dy},$$

$$\bar{H}_0 = \frac{1}{4\pi} \iiint \frac{H_2}{r} \, dx'dy'dz' - \frac{dX_0}{dz},$$

and, in general,

$$F_n = \frac{1}{4\pi} \iiint \frac{F_{n+2}}{r} \, dx'dy'dz',$$

$$G_n = \frac{1}{4\pi} \iiint \frac{G_{n+2}}{r} \, dx'dy'dz',$$

$$H_n = \frac{1}{4\pi} \iiint \frac{H_{n+2}}{r} \, dx'dy'dz'.$$

From these, by differentiation, we have, after changing $n + 1$ to n,

$$F_n = \frac{1}{4\pi} \iiint \left\{ H_{n+1} \frac{d\frac{1}{r}}{dy} - G_{n+1} \frac{d\frac{1}{r}}{dz} \right\} dx'dy'dz',$$

$$G_n = \frac{1}{4\pi} \iiint \left\{ F_{n+1} \frac{d\frac{1}{r}}{dz} - H_{n+1} \frac{d\frac{1}{r}}{dx} \right\} dx'dy'dz',$$

$$H_n = \frac{1}{4\pi} \iiint \left\{ G_{n+1} \frac{d\frac{1}{r}}{dx} - F_{n+1} \frac{d\frac{1}{r}}{dy} \right\} dx'dy'dz'.$$

Let us now write

$$F_n = \bar{F}_n + \frac{d\chi_n}{dx},$$

$$G_n = \bar{G}_n + \frac{d\chi_n}{dy},$$

$$H_n = \bar{H}_n + \frac{d\chi_n}{dz},$$

where

$$\chi_n = \frac{1}{4\pi}\iiint \frac{1}{r}\left\{\frac{d\bar{F}_n}{dx} + \frac{d\bar{G}_n}{dy} + \frac{d\bar{H}_n}{dz}\right\} dx'dy'dz'.$$

For all points where $\bar{F}_n = \bar{G}_n = \bar{H}_n = 0$, $-\chi_n$ will be the potential of F_n, G_n, H_n. From the form of these equations, F_n, G_n, and H_n satisfy the equation of continuity, while $\bar{F}_n$, $\bar{G}_n$, and $\bar{H}_n$ do not. *Hence every discontinuous system of vectors, $\bar{F}_n$, $\bar{G}_n$, $\bar{H}_n$, can be made continuous by the addition of the derivatives of a certain quantity having the value*

$$\frac{1}{4\pi}\iiint \frac{1}{r}\left\{\frac{d\bar{F}}{dx} + \frac{d\bar{G}}{dy} + \frac{d\bar{H}}{dz}\right\} dxdydz$$

and the new system so formed satisfies the equation of continuity.

To find a discontinuous system from a continuous one, we can draw surfaces which divide space into acyclic regions with respect to the kind of motion, and the discontinuous system will be distributed over these surfaces.

If there is some necessary physical condition by which the vectors *must* satisfy the equation of continuity, then the discontinuous* system of vectors $\bar{F}_n$, $\bar{G}_n$, $\bar{H}_n$ will evidently have the same effect in every way as the continuous system F_n, G_n, and H_n, so that one system implies the existence of the other.

This theorem is at the basis of all theorems of such a nature as those which express the energy of a magnetic system either by an integral throughout space, or by one throughout the magnets, or by one over the surface of the magnets.

The value of χ_n can be expressed otherwise, as in Maxwell's Electricity, Art. 385, where it is shown that the integral can be divided into two integrals, one a surface integral and the other a volume integral. In comparing my formulæ with Maxwell's, however, it must not be forgotten that my volume integrals are to be taken throughout the *whole of space*, whereas Maxwell's gen-

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* In this paper I nearly always use the term "continuous system of vectors" to indicate a system which satisfies the equation of continuity.

But a distribution of vectors can evidently have a *side* discontinuity as well as an end discontinuity. The context will always indicate my meaning, especially as I use the dash over quantities which have *end* discontinuity and so do not satisfy the equation of continuity.

erally refer to the interior of some surface. Hence, in the case referred to, the surface is at an infinite distance and the two volume integrals are equal.

Consider the integral

$$\chi_n = \frac{1}{4\pi}\iiint \frac{1}{r}\left\{\frac{d\bar{F}_n}{dx} + \frac{d\bar{G}_n}{dy} + \frac{d\bar{H}_n}{dz}\right\} dxdydz$$

taken without exception throughout space.

This is equal to the sum of a surface and volume integral: but if we take the surface where $\bar{F}_n$, $\bar{G}_n$, and $\bar{H}_n$ are zero, the surface integral becomes zero, and leaves

$$\chi_n = -\frac{1}{4\pi}\iiint \left\{\bar{F}_n \frac{d\frac{1}{r}}{dx} + \bar{G}_n \frac{d\frac{1}{r}}{dy} + \bar{H}_n \frac{d\frac{1}{r}}{dz}\right\} dxdydz,$$

or we may write it

$$\chi_n = -\frac{1}{4\pi}\iiint \frac{\bar{M}_n \cos\epsilon}{r^2} dxdydz,$$

where $\bar{M}_n$ is the resultant vector of the n^{th} order, and ϵ is the angle between it and r.

If the point is within the region where $\bar{F}_n$, $\bar{G}_n$, $\bar{H}_n$ exist, the integral will vanish at the lower limit, $r = 0$, and so the value can still be used within such space. For a region within which

$$\frac{d\bar{F}_n}{dx} + \frac{d\bar{G}_n}{dy} + \frac{d\bar{H}_n}{dz} = 0$$

the integral is much simplified. For it is only at the surface that this quantity has a value. Let the surface consist of an infinitely thin region of thickness $d\nu$, within which $\bar{M}_n$ decreases uniformly from its value inside to zero, then

$$\frac{d\bar{F}_n}{dx} = -\frac{d\bar{F}_n}{d\nu}\frac{d\nu}{dx},$$

$$\frac{d\bar{G}_n}{dy} = -\frac{d\bar{G}_n}{d\nu}\frac{d\nu}{dy},$$

$$\frac{d\bar{H}_n}{dz} = -\frac{d\bar{H}_n}{d\nu}\frac{d\nu}{dz}.$$

And since $dxdydz = d\nu dS$, we have

$$\chi_n = -\frac{1}{4\pi}\iint \frac{\bar{\phi}_n \cos\theta}{r^2} dS,$$

where $\bar{\phi}_n$ is the potential of $\bar{F}_n$, $\bar{G}_n$, $\bar{H}_n$, and θ is the angle between the radius vector, r, and the normal drawn outwards from the surface, and the inte-

gration is over the surface. Multiplying the equations by $\frac{1}{r}\,dxdydz$, and integrating, we see that we can write

$$F_n = \frac{1}{4\pi}\iiint \frac{\overline{F}_{n+2}}{r}\,dx'dy'dz' - \frac{d\xi_{n+2}}{dx},$$

$$G_n = \frac{1}{4\pi}\iiint \frac{\overline{G}_{n+2}}{r}\,dx'dy'dz' - \frac{d\xi_{n+2}}{dy},$$

$$H_n = \frac{1}{4\pi}\iiint \frac{\overline{H}_{n+2}}{r}\,dx'dy'dz' - \frac{d\xi_{n+2}}{dz}.$$

By differentiation and replacement of $n + 1$ by n, we have

$$F_n = \frac{1}{4\pi}\iiint\left\{\overline{H}_{n+1}\frac{d\frac{1}{r}}{dy} - \overline{G}_{n+1}\frac{d\frac{1}{r}}{dz}\right\}dxdydz - \frac{d}{dx}\left\{\frac{d\xi_{n+1}}{dy} - \frac{d\xi_{n+1}}{dz}\right\},$$

$$G_n = \frac{1}{4\pi}\iiint\left\{\overline{F}_{n+1}\frac{d\frac{1}{r}}{dz} - \overline{H}_{n+1}\frac{d\frac{1}{r}}{dx}\right\}dxdydz - \frac{d}{dy}\left\{\frac{d\xi_{n+1}}{dz} - \frac{d\xi_{n+1}}{dx}\right\},$$

$$H_n = \frac{1}{4\pi}\iiint\left\{\overline{G}_{n+1}\frac{d\frac{1}{r}}{dx} - \overline{F}_{n+1}\frac{d\frac{1}{r}}{dy}\right\}dxdydz - \frac{d}{dz}\left\{\frac{d\xi_{n+1}}{dx} - \frac{d\xi_{n+1}}{dy}\right\},$$

where the value of ξ_{n+2} is

$$\xi_{n+2} = -\frac{1}{4\pi}\iiint \frac{\chi_{n+2}}{r}\,dxdydz$$

or $$\xi_{n+2} = -\frac{1}{16\pi^2}\iiint\iiint\frac{1}{Rr}\left\{\frac{d\overline{F}_{n+2}}{dx} + \frac{d\overline{G}_{n+2}}{dy} + \frac{d\overline{H}_{n+2}}{dz}\right\}dxdydz\,dx'dy'dz'.$$

One of these integrations is to be taken throughout all space, supposing $\overline{F}_{n+2}$, $\overline{G}_{n+2}$, $\overline{H}_{n+2}$ constant, and the other throughout space, supposing them variable.

The first can be performed, for it is simply

$$\iiint\frac{1}{Rr}\,dxdydz;$$

but we know that

$$\Delta^2 r = \frac{2}{r},$$

whence we may write

$$\iiint \frac{1}{Rr}\,dxdydz = -2\pi r.$$

These letters are connected by the relation that the R and r of the left-hand member and the r of the right-hand one form a triangle, the element $dxdydz$ being at the intersection of the R and r of the left-hand member. The integral vanishes for the limit $R = 0$.

Hence $$\xi_{n-2} = \frac{1}{8\pi}\iiint r\left\{\frac{d\bar{F}_{n-2}}{dx'} + \frac{d\bar{G}_{n-2}}{dy'} + \frac{d\bar{H}_{n-2}}{dz'}\right\} dx'dy'dz'.$$

This can be put in the form

$$\xi_{n-2} = -\frac{1}{8\pi}\iiint \bar{M}_{n-2}\cos\epsilon\, dxdydz,$$

where ϵ is the angle between $\bar{M}_n$ and the radius vector r.

We can now develop the following general method of finding the lower vectors from the higher. Let us write

$$O_n = \frac{1}{2}\iiint \bar{F}_n r dx'dy'dz',$$

$$P_n = \frac{1}{2}\iiint \bar{G}_n r dx'dy'dz',$$

$$Q_n = \frac{1}{2}\iiint \bar{H}_n r dx'dy'dz'.$$

Then we can write

$$\xi_n = -\frac{1}{4\pi}\left(\frac{dO_n}{dx} + \frac{dP_n}{dy} + \frac{dQ_n}{dz}\right) = \frac{1}{4\pi}\left(\frac{dO_n}{dx'} + \frac{dP_n}{dy'} + \frac{dQ_n}{dz'}\right),$$

and the equations become

$$F_{n-2} = \frac{1}{4\pi}\left\{\Delta^2 O_n + \frac{d}{dx}\left(\frac{dO_n}{dx} + \frac{dP_n}{dy} + \frac{dQ_n}{dz}\right)\right\},$$

$$G_{n-2} = \frac{1}{4\pi}\left\{\Delta^2 P_n + \frac{d}{dy}\left(\frac{dO_n}{dx} + \frac{dP_n}{dy} + \frac{dQ_n}{dz}\right)\right\},$$

$$H_{n-2} = \frac{1}{4\pi}\left\{\Delta^2 Q_n + \frac{d}{dz}\left(\frac{dO_n}{dx} + \frac{dP_n}{dy} + \frac{dQ_n}{dz}\right)\right\}.$$

Let us now see what will happen if we multiply these by $\frac{1}{4\pi R}dxdydz$ and integrate throughout space, R being a radius vector.

The first terms become F_{n-4}, G_{n-4}, and H_{n-4}, and we only have to consider the integrals of O_n, P_n, Q_n. We have to obtain an integral of the form

$$\iiint \frac{f(xyz)\phi(r)}{R}\, dxdydz,\ dx'dy'dz'.$$

Here we note that the radius vector r is the distance between the two points x, y, z and x', y', z', and so the integral reduces to

$$\iiint f(x'y'z')\iiint\left\{\frac{\phi(r)}{R}\, dxdydz\right\} dx'dy'dz'.$$

In this case, to obtain the integral with respect to r we make use of the fact that

$$\Delta^2 r^n = n(n+1)\,r^{n-2},$$

and we thus find

$$\frac{1}{4\pi}\iiint \frac{r^n}{R}\,dxdydz = -\frac{1}{(n+2)(n+3)}\,r^{n+2}.$$

The integral extends throughout space, and the r and R of the first member, and the r of the second member form a triangle, the point x, y, z being at their intersection. Hence,

$$\Delta^2 \iiint f(x', y', z')\,r^n dx'dy'dz' = n(n+1)\iiint f(x', y', z')\,r^{n-2}dx'dy'dz'.$$

We have thus found F_{n-4}, G_{n-4}, H_{n-4}, and by repeating the operation s times, counting the original operation as one, we can find F_{n-2s}, G_{n-2s}, H_{n-2s}, and F_{n-2s+1}, G_{n-2s+1}, H_{n-2s+1} by differentiation of these. The result is

$$O_{n,s} = \frac{(-1)^{s+1}}{1,2,3\ldots.2s}\iiint \bar{F}_n r^{2s-1}\,dxdydz,$$

$$P_{n,s} = \frac{(-1)^{s+1}}{1,2,3\ldots.2s}\iiint \bar{G}_n r^{2s-1}\,dxdydz,$$

$$Q_{n,s} = \frac{(-1)^{s+1}}{1,2,3\ldots.2s}\iiint \bar{H}_n r^{2s-1}\,dxdydz.$$

For the case $s = 0$, the coefficient of the integral becomes simply -1 and $\xi_{n,o}$ becomes $-\chi_n$. We also observe that

$$O_{n(s-t)} = (\Delta^2)^t(O_{n,s}),$$

$$P_{n(s-t)} = (\Delta^2)^t(P_{n,s}),$$

$$Q_{n(s-t)} = (\Delta^2)^t(Q_{n,s}),$$

and

$$\xi_{n,s} = -\frac{1}{4\pi}\left(\frac{dO_{n,s}}{dx} + \frac{dP_{n,s}}{dy} + \frac{dQ_{n,s}}{dz}\right),$$

$$\xi_{n(s-t)} = (\Delta^2)^t \xi_{n,s}.$$

In these the symbol $(\Delta^2)^t$ signifies that the operation Δ^2 is to be repeated t times. Hence we may write, in general,

$$F_{n-2s} = \frac{O_{n,s-1}}{4\pi} - \frac{d\xi_{n,s}}{dx},$$

$$G_{n-2s} = \frac{P_{n,s-1}}{4\pi} - \frac{d\xi_{n,s}}{dy},$$

$$H_{n-2s} = \frac{Q_{n,s-1}}{4\pi} - \frac{d\xi_{n,s}}{dz},$$

and

$$F_{n-2s-1} = \frac{dH_{n-2s}}{dy} - \frac{dG_{n-2s}}{dz},$$

$$G_{n-2s-1} = \frac{dF_{n-2s}}{dz} - \frac{dH_{n-2s}}{dx},$$

$$H_{n-2s-1} = \frac{dG_{n-2s}}{dx} - \frac{nF_{n-2s}}{dy}.$$

Thus we have expressed the whole series of vectors in terms of any one series, and have shown how to calculate the distribution of them all from the discontinuous distribution of any one order, and this by a single set of integrals, with the proper differentiation.

The value of $\xi_{n,s}$ can be expressed simply by the integral

$$\xi_{n,s} = \frac{1}{4\pi} \frac{(-1)^{s-1}}{1,2.3\ldots\ldots 2s} \iiint \bar{M}_n r^{2s-2} \cos\epsilon \, dxdydz,$$

where ϵ is the angle between $\bar{M}_n$ and r. For all space in which $\bar{M}_n$ is zero, F_n, G_n, H_n have a potential found by making $s = 0$, whence

$$\xi_{n.0} = -\chi_n = \frac{1}{4\pi} \iiint \frac{\bar{M}_n \cos\epsilon}{r^2} \, dxdydz.$$

$\xi_{n,0}$ satisfies Laplace's equation at all points where $\bar{M}_n = 0$, and this potential can be considered as due either to the distribution of $\bar{M}_n$ within a surface, or of M_{n-1} over the surface, the components of M_{n-1} satisfying the equation of continuity, but those of $\bar{M}_n$ not.

Indeed, we can consider *any* of the systems of vectors as the *cause* and all the others as the *effect*.

When $\bar{F}_n, \bar{G}_n, \bar{H}_n$ satisfy the equation of continuity within a surface, the integrals reduce to the surface integral of the next higher order of vectors.

But in the above operation of finding $F_{n-2s}, G_{n-2s}, H_{n-2s}$ from $\bar{F}_n, \bar{G}_n, \bar{H}_n$, we have neglected the constants of integration, and we have now to consider their effect.

We first observe that, as the higher orders are obtained from the lower by differentiating, they are perfectly determined by these equations. But as we

have to integrate in obtaining the lower orders from the higher, the result requires examination. Taking the equations for F_n, G_n, H_n, and adding to them respectively π, π_1, π_{11}, we have

$$F_n = \frac{1}{4\pi}\iiint \frac{F_{n+2}}{r}\, dx'dy'dz' + \pi,$$

$$G_n = \frac{1}{4\pi}\iiint \frac{G_{n+2}}{r}\, dx'dy'dz' + \pi_1,$$

$$H_n = \frac{1}{4\pi}\iiint \frac{H_{n+2}}{r}\, dx'dy'dz' + \pi_{11}.$$

Performing the operation Δ^2 on these, with respect to the point x', y', z', and also noting that F_n, G_n, H_n satisfy the equation of continuity, we shall have

$$\Delta^2\pi = 0, \qquad \Delta^2\pi_1 = 0, \qquad \Delta^2\pi_{11} = 0,$$

$$\frac{d\pi}{dx} + \frac{d\pi_1}{dy} + \frac{d\pi_{11}}{dz} = 0.$$

Now these quantities must not affect the values of F_{n+1}, G_{n+1}, H_{n+1} as obtained from F_n, G_n, H_n, and so we must also have

$$\frac{d\pi_{11}}{dy} - \frac{d\pi_1}{dz} = 0,$$

$$\frac{d\pi}{dz} - \frac{d\pi_{11}}{dx} = 0,$$

$$\frac{d\pi_1}{dx} - \frac{d\pi}{dy} = 0.$$

The solution of these equations is found in the values

$$\pi = \frac{dV}{dx}, \qquad \pi_1 = \frac{dV}{dy}, \qquad \pi_{11} = \frac{dV}{dz},$$

where V must satisfy Laplace's equation throughout all space. And a term of this kind is the proper thing to add, no matter how many integrations are performed. But if this term satisfies Laplace's equation throughout all space, the *cause* of the potential V must be at an infinite distance. As this quantity is entirely arbitrary, and has no relation to the original system of vectors which we consider as the *cause* of the vectors of the lower order, we may generally make it zero unless there is some other condition to be satisfied. But the integrals without any addition give us all the lower orders of vectors which are dependent on the higher. Hence no addition should be made to the integrals when we are searching for the effect of a cause.

Indeed, all such cases of a potential satisfying Laplace's equation throughout space can always be represented by a distribution over a sphere at an infinite distance, and in Physics it must always be looked upon in that way. In fluid motion the only quantities which we can add without destroying the original configuration is a general motion of translation and rotation of the fluid represented by the equations

$$\pi = A + Bx + Cy + Dz,$$
$$\pi_1 = A_1 + B_1x + C_1y + D_1z,$$
$$\pi_{11} = A_{11} + B_{11}x + C_{11}y + D_{11}z,$$

where we have the relations

$$B + C_1 + D_{11} = 0; \ C_{11} = D_1; \ D = B_{11}; \ C = B_1.$$

We can define V by saying that it is a solid harmonic of a positive degree, so that it becomes infinite at an infinite distance.

We see from these facts that if any quantities, F_n, G_n, H_n, vanish, then all the others must have the values given by the above equations.

Hence all the systems of vectors must exist somewhere in space, though it is evident that there may be regions where some of the orders vanish. When this is the case, and throughout the region the vectors of the n^{th} order vanish, then all orders of vectors greater than the n^{th} vanish also, throughout the same space, but not those less than this; and the vectors of the $(n-1)^{th}$ order have a potential satisfying Laplace's equation.

Furthermore, if within any surface we have a discontinuous distribution of the n^{th} order of vectors, then we have seen that without the surface the vectors F_n, G_n, H_n have a potential satisfying Laplace's equation, and consequently all vectors above the n^{th} order will vanish outside that surface.

If within or over a given surface the vectors F_n, G_n, H_n form closed circuits, then the vectors F_n, G_n, H_n, and of higher orders due to the above vectors will be zero without the surface, and the vectors of the $(n-1)^{th}$ order will have a potential. But in the surface, or without it, all the vectors exist. Conversely, if throughout any space the vectors of the n^{th} order have a potential satisfying Laplace's equation, the vectors of the $(n-1)^{th}$ order will be zero in that space.

If within any surface we have a discontinuous distribution of the n^{th} order of vectors, it is evident that the external effect will be the same as the continuous distribution within the same surface of the $(n-1)^{th}$ or higher orders. Indeed, the effect will be the same throughout space.

We have seen that there are two methods of looking at a series of vectors,— one as if they were derived from those above, and the other from those below. Now these methods are similar to the direct and inverse methods of treating electrical problems. The inverse method invented by Green is by far the most useful in electricity, and it seems to me that a similar method will be useful in the treatment of vector quantities. Indeed, the method leads to most important results. Consider the equations

$$F_{n+1} = \frac{dH_n}{dy} - \frac{dG_n}{dz},$$

$$G_{n+1} = \frac{dF_n}{dz} - \frac{dH}{dx},$$

$$H_{n+1} = \frac{dG_n}{dx} - \frac{dF}{dy}.$$

The operations here indicate that we choose the vectors of the n^{th} order, and find their *cause* in the vectors of the $(n+1)^{\text{th}}$ order; whereas the direct operation assumes the *cause* and finds the *effect*. This is similar to the case of electricity where the *cause* of the potential V is discovered by the operation $-\frac{1}{4\pi}\Delta^2$ to be the density of electricity ρ, with its proper distribution. And just as we find many electrical distributions over surfaces which have the same external effect, so we are able to find many distributions of the vectors F_{n+1}, G_{n+1}, H_{n+1}, over surfaces which produce the same external values of F_n, G_n, and H_n.

Suppose that within a surface F_n, G_n, H_n have a potential $\xi_{n,0}$, and outside a potential $\xi'_{n,0}$, both of which satisfy Laplace's equation. To satisfy the equation of continuity we must have at the surface,

$$\frac{d\xi_{n.0}}{d\nu} = \frac{d\xi'_{n.0}}{d\nu}.$$

Our equations then show that F_{n+1}, G_{n+1}, H_{n+1} are zero throughout all space except the surface. At the surface the equations become

$$F_{n+1}\,d\nu = \left(\frac{d\xi'_{n.0}}{dz} - \frac{d\xi_{n.0}}{dz}\right)\frac{d\nu}{dy} - \left(\frac{d\xi'_{n.0}}{dy} - \frac{d\xi_{n.0}}{dy}\right)\frac{d\nu}{dz},$$

$$G_{n+1}\,d\nu = \left(\frac{d\xi'_{n.0}}{dx} - \frac{d\xi_{n.0}}{dx}\right)\frac{d\nu}{dz} - \left(\frac{d\xi'_{n.0}}{dz} - \frac{d\xi_{n.0}}{dz}\right)\frac{d\nu}{dx},$$

$$H_{n+1}\,d\nu = \left(\frac{d\xi'_{n.0}}{dy} - \frac{d\xi_{n.0}}{dy}\right)\frac{d\nu}{dx} - \left(\frac{d\xi'_{n.0}}{dx} - \frac{d\xi_{n.0}}{dx}\right)\frac{d\nu}{dy}.$$

This distribution of F_{n-1}, G_{n-1}, H_{n-1} will produce the potentials $\xi_{n,i}$ and $\xi'_{n,o}$, the one within and the other without the surface. The most interesting case of this discontinuity at a surface is found by making

$$\frac{d\xi_{n,i}}{d\nu} = 0 \quad \text{and} \quad \frac{d\xi'_{n,o}}{d\nu} = 0$$

at the surface, in which case the surface is a stream surface for both $\xi'_{n,o}$ and $\xi_{n,i}$, and the equation of continuity for the n^{th} order is satisfied.

Let us now make $\xi_{n,i}$ = constant, and we have

$$F_{n-1}\,d\nu = \frac{d\xi'_{n,o}}{dz}\frac{d\nu}{dy} - \frac{d\xi'_{n,o}}{dy}\frac{d\nu}{dz},$$

$$G_{n-1}\,d\nu = \frac{d\xi'_{n,o}}{dx}\frac{d\nu}{dz} - \frac{d\xi'_{n,o}}{dz}\frac{d\nu}{dx},$$

$$H_{n-1}\,d\nu = \frac{d\xi'_{n,o}}{dy}\frac{d\nu}{dx} - \frac{d\xi'_{n,o}}{dx}\frac{d\nu}{dy}.$$

which will apply to all surfaces as well as stream surfaces.

If we had made $\xi'_{n,o}$ = constant instead of $\xi_{n,i}$, we should have obtained equations of the same form, but with the opposite sign. If, however, we say that the normal shall be drawn from the side of the surface containing the original source of potential to the other side, the sign will be the same. Thus our surface must be drawn so as to separate space into two parts, one of which contains the source, and our equations give us the values to distribute over the surface, so that, on the side *opposite* the source, the effect will be the same as from the original source. Hence, although we may express a proposition with respect to the *outside* and *inside*, yet we may always reverse the terms into *inside* and *outside*. This is exactly analogous to the proposition with regard to the distribution of electricity. If we distribute the vectors over the surface with a negative sign, they will just neutralize the effect of the original source on the side opposite to it, which is exactly analogous to the case of electrostatic induction.

As to the total amount of the vector M_{n-1} to be distributed over the surface, it is evident that the *quantity* in the two systems must be the same.

I here use the term quantity in the sense of surface integral across the section of the vector. Hence, if M'_{n-1} is the original and M_{n-1} the new value of the vector, we must have

$$\iint M'_{n-1}\,dS' = \iint M_{n-1}\,dS,$$

the integral to be taken over a surface cutting the vectors at right angles.

Having chosen our stream surface of the n^{th} order, we now have to define the direction of M_{n+1} on the surface. The equations express the fact that $M_{n+1}dv$ is in the direction of the intersection of the given surface with the equipotential surface, and has the value M_n. The equipotential surfaces and two systems of stream surfaces can be found so as to form an orthogonal system. But a stream surface in general is not necessarily one of them, but is only a surface containing the stream lines or lines of direction of M_n. When we choose for our surface one of the stream surfaces of the orthogonal system, the intersection of the two stream surfaces is an equipotential line for M_{n+1}, and the intersection of the stream surface and an equipotential surface is a stream line for M_{n+1}. We have seen that the potential ξ'_n of the n^{th} order can arise either from a distribution of vectors of the $(n+1)^{\text{th}}$ order in closed circuits or a discontinuous distribution of the n^{th} order; and that the $(n+1)^{\text{th}}$ order was distributed in closed lines around the n^{th} order. The relations are similar to those between magnetism and electricity and between a magnetic shell and a current around its edge. In this case we can obtain a distribution of a shell of $\bar{M}_n$ over the surface which is equivalent to the distribution of M_{n+1} over the surface. The strength of the shell, $\bar{M}_n dv$, will evidently be found by adding all the shells together, and hence, if we draw any line l on the surface,

$$\bar{M}_n dv = \int M_{n+1} dv \cos\theta \, dl,$$

but this is equivalent to

$$\bar{M}_n dv = \int M_n dv \cos\theta \, dl = \xi_{n,0} + \text{constant},$$

because $M_{n+1}dv$ is equal to M_n at the surface.

Hence the strength of the shell is equal to the potential. Knowing $M_{n+1}dv$ originally, we can thus find $\bar{M}_n dv$, whence the potential is

$$\xi_{n,0} = \frac{1}{4\pi} \iint \frac{\bar{M}_n dv \cos\epsilon}{r^2} \, dS,$$

ϵ being the angle between r and the *normal* to the surface. This integral must not only be taken over the surface which we have been considering, but also over diaphragms which we must draw so as to convert cyclic space into acyclic.

Thus the theorem which I have just given merely shows us how to draw surfaces so that this integral taken over them shall be the same for all regions which can be reached without passing through a surface.

It has already been known that the diaphragms could be drawn in any position, but this theorem applies also to the other positions of the surface boundary of the space under consideration.

We may evidently have two varieties of surfaces which divide the internal space into cyclic or acyclic regions. The first are obtained by forming, as it were, tubes around the closed circuits of the vector M_{n+1}, which we can consider as the original source of $\xi'_{n,0}$. Distributing M_n over this surface in the manner described, and removing the original source, the new system has the same external effect as the old, but makes $\xi_{n,0}$ constant inside. The second variety of surface is acyclic inside, and encloses the original distribution of M_{n+1} as a whole. It is evident that this last variety of surface cannot be a stream surface for $\xi'_{n,0}$, unless there is some other source of M_n outside the surface. Our process, then, merely replaces one of these sources for the points on the other side of the surface.

Taking any cyclic value of the potential $\xi'_{n,0}$ and drawing a tube of flow of any cross section, and applying this method to it, the distribution of M_{n+1} over the surface will cause the potential inside it to become constant. Take away the external system now, and we shall have a potential equal to $-\xi'_{n,0}$ inside the tube, and constant without.

If we take any two of our surfaces and distribute $+M_{n+1}$ over the internal one, and $-M_{n+1}$ over the external one, according to the equations, then within the inside surface we shall have $\xi_{n,0}$ constant; between the two surfaces $\xi'_{n,0}$ has its former value; and outside the outer one $\xi_{n,0}$ is constant once more. In fluid motion this will give us the case of a mass of liquid revolving in a liquid at rest, with a core of liquid at rest.

If we allow the original distribution to remain, and distribute $-M_{n+1}$ over the surface, then the potential *inside* remains the same and becomes constant *outside*. Thus the inside and outside of the surface are reciprocal, and we can always reverse the statement in this manner.

Helmholtz has considered some cases of discontinuous fluid motion, and other writers have occupied themselves more or less with it. But most, if not all of them have considered the vortex surface between the two moving fluids as the *effect*, whereas I here treat the vortices as the *cause* of the fluid motion, and have thus obtained a method of replacing the original vortices by one or more vortex sheets around them. This is the basis of the inverse method of treating hydrodynamics.

To get a solid distribution of M_{n+1}, we can place one surface within the other with its proper distribution so as to make what is required. If within a given surface any system of vectors are distributed so as to have a potential within that surface, then the continuous system as obtained from this system will have a potential both within and without the surface, but the potential inside will not

be the same as that of the discontinuous system, but will be that minus $\xi_{n,0}$. At the surface of discontinuity all the higher system of vectors will be collected.

Let us suppose that we have the potential $\xi_{n,0}$ within the surface, and $\xi'_{n,s}$ outside the surface, both of which satisfy Laplace's equation for the space in which they are used. We can get the conditions at any surface, but as they are complicated, let us take the surfaces, stream surfaces for both functions.

Then at the surface

$$\frac{d\xi_{n.0}}{d\nu} = 0; \ \frac{d\xi'_{n.s}}{d\nu} = 0.$$

According to our notation, $\xi_{n,0}$ is the potential of a *higher* order of motion than $\xi'_{n,s}$, the first being the potential of F_n, G_n, H_n and the last of F_{n-2s}, G_{n-2s}, H_{n-2s}.

If we should distribute $-F_{n+1}$, $-G_{n+1}$, $-H_{n+1}$, over the surface according to the previous investigation, we should have $\xi_{n,0} = \xi_{n,0}$ inside, and constant outside. If F_{n-2s+1}, G_{n-2s+1}, H_{n-2s+1}, were distributed over the surface in the proper manner, we should have $\xi'_{n,s} = \xi'_{n,s}$ outside and constant inside. If the two existed together, there would be

$$\xi_{n,0} + \text{constant inside},$$

and

$$\xi'_{n,s} + \text{constant outside}.$$

So that in this case we must distribute over the surface

$$-F_{n+1} d\nu = -\frac{d\xi_{n.0}}{dz}\frac{d\nu}{dy} + \frac{d\xi_{n.0}}{dy}\frac{d\nu}{dz},$$

$$-G_{n+1} d\nu = -\frac{d\xi_{n.0}}{dx}\frac{d\nu}{dz} + \frac{d\xi_{n.0}}{dz}\frac{d\nu}{dx},$$

$$-H_{n+1} d\nu = -\frac{d\xi_{n.0}}{dy}\frac{d\nu}{dx} + \frac{d\xi_{n.0}}{dx}\frac{d\nu}{dy},$$

and

$$F_{n-2s+1} d\nu = \frac{d\xi'_{n.s}}{dz}\frac{d\nu}{dy} - \frac{d\xi'_{n.s}}{dy}\frac{d\nu}{dz},$$

$$G_{n-2s+1} d\nu = \frac{d\xi'_{n.s}}{dx}\frac{d\nu}{dz} - \frac{d\xi'_{n.s}}{dz}\frac{d\nu}{dx},$$

$$H_{n-2s+1} d\nu = \frac{d\xi'_{n.s}}{dy}\frac{d\nu}{dx} - \frac{d\xi'_{n.s}}{dx}\frac{d\nu}{dy}.$$

Gauss's theorem applied to electricity gives us the amount of electricity in a surface, from the surface integral of the electric force. Let us now attempt to develop some similar method for this case.

Taking Poisson's equation,

$$4\pi\rho = -\Delta^2 V,$$

Gauss's theorem gives

$$\iint \frac{dV}{dv}\, dS = -4\pi \iiint \rho\, dxdydz,$$

where the first integral extends over a surface, and the second over the space included within the surface. If we have a distribution of F_n, G_n, H_n, within a given surface, then F_{n+2}, G_{n+2}, H_{n+2}, are obtained from them by the equations

$$F_{n+2} = -\Delta^2 F_n,$$
$$G_{n+2} = -\Delta^2 G_n,$$
$$H_{n+2} = -\Delta^2 H_n.$$

Hence we can write immediately

$$\iint \frac{dF_n}{dv}\, dS = -\iiint F_{n+2}\, dxdydz,$$
$$\iint \frac{dG_n}{dv}\, dS = -\iiint G_{n+2}\, dxdydz,$$
$$\iint \frac{dH_n}{dv}\, dS = -\iiint H_{n+2}\, dxdydz,$$

As we have

$$J_n = -\Delta^2 \chi_n = \Delta^2 \xi_{n,0},$$

therefore we can also write

$$\iint \frac{d\xi_{n,0}}{dv}\, dS = -\iiint J_n\, dxdydz,$$

where

$$J_n = \frac{d\bar{F}_n}{dx} + \frac{d\bar{G}_n}{dy} + \frac{d\bar{H}_n}{dz}.$$

To get a clear idea of the meaning of J_n, I may remark that in the case of magnetism it gives us the distribution of the so called magnetic matter. In the present case the integral vanishes over any surface enclosing the original source, so that the algebraical sum of J_n within the surface is zero, or the poles of the magnet are of equal strength.

I have thus developed the general theory of vectors from a purely mathematical point of view, so that we may apply it to any subject involving these important quantities, and thus many excellent methods for use in electromagnetism may be found. There are even some methods which give important

results in the theory of attraction and spherical harmonics, especially the theorem that

$$\Delta^2 \iiint f(x', y', z',)\, r^n dx'dy'dz' = n(n+1) \iiint f(x', y', z',)\, r^{n-2} dx'dy'dz',$$

Δ^2 being taken with respect to x, y, z, and not with respect to x', y', z'. This gives the potential with a force varying as r^{n-3} from that varying as r^{n-1}.

But the most important and interesting application is to hydrodynamics, and I shall then develop a few more points which might have been placed above.

There is an important point connected with this theory which will be extremely useful to us further on; and that is that where M_n exists and varies from one point to another, then we shall sometimes be obliged to use an element of the fluid in our calculations for the determination of M_0 infinitely smaller than for M_n. In other words, M_0 will not be determined exactly within an element without going still lower and assuming something further within the element. Thus, in case we have an infinitely small tube having the vector M_n in the direction of its length, we find that there are an infinite number of ways in which the vector can be distributed within the tube and yet produce the same external effect. This may be unimportant in some cases, but a neglect of this principle will cause many interesting problems in hydrodynamics to assume an indeterminate form. Thus the problem of the velocity of a vortex ring needs to be calculated in this way. Assuming the distribution of finer vortex filaments within the substance of the vortex, and the problem becomes determinate. I shall thus show that the vortex ring moves according to that distribution.

The Kinematics of Fluid Motion.

The fluid is considered as unlimited and without solid bodies in it. The motion of a fluid is a vector quantity, and so the above theory must apply.

Let F_0, G_0, H_0, be the components of the fluid velocity,* and let the fluid be incompressible. Then we have

$$\frac{dF_0}{dx} + \frac{dG_0}{dy} + \frac{dH_0}{dz} = 0.$$

The vectors of the higher order, which I shall for simplicity call motions, must then be interpreted. The motion of the first order whose components are

* We could evidently take any other order for the fluid velocity, and, in any problem, we can generally change the suffixes to the letters, and thus obtain new solutions.

F_1, G_1, H_1, has been interpreted by Helmholtz to be vortex motion, and it remains to interpret the rest.

The second order will be, as it were, the vortices of the vortices, but to interpret it better I shall proceed as follows: —

Suppose that along the axis of x there is a line of motion of the n^{th} order. Calculate from this the values of F_4, G_4, and H_4, and we shall have the true motion of the fluid. It will be found that in all motions of the even degrees the fluid moves parallel to the axis of x, but in motions of the odd degree the true motion of the fluid is in circles around the axis. Hence the motions of the even degrees are translatory motions and those of the odd degrees are vortical motions.

Motion of the zero order is ordinarily irrotational motion.

Motion of the first order is the vortex motion of Helmholtz, and a line of motion of the first order is a vortex filament.

Motion of the second order is what I have, in a previous paper, called the *relative* motion of the fluid, and a line of this motion has the fluid flowing forward along its axis with very great velocity and a less and less velocity as we proceed outward from the axis, until we reach a certain distance, where it is zero; beyond this the velocity increases as we pass from the axis, and is infinite at an infinite distance in the impossible case of the line of motion being straight and infinitely long. In the possible cases where the line is closed, the motion of the fluid is finite everywhere, and is zero at infinity.

Motion of the third order can be roughly conceived of as one vortex filament within another revolving in the opposite direction.

Motion of the fourth order can be roughly conceived of as one line of motion of the second order within another larger and opposite one.

And so, as we go upward, an element of the line of motion becomes more and more complicated. The motion of the fluid exterior to the line of motion of any order can be calculated by formulæ as simple for high as for low orders of motion, though the integrations are generally somewhat more difficult as we go higher.

Mathematically there are motions of the negative orders also, and they are useful in some calculations; but the physical conception does not at first sight lead to such important results as the positive, though I hope to investigate this further.

Let us now conceive of a closed line in space of any form whatever having a uniform distribution of one of the higher motions along it.

As the curve is closed such a distribution will satisfy the equation of continuity.

If the motion along the line is of the n^{th} order, then there will be a distribution throughout space of motion of the $(n-1)^{th}$ order and below, but no motion of the n^{th} or higher orders. And the motion of the fluid in this case will be equivalent to that produced by a discontinuous but uniform distribution of motion of the $(n-1)^{th}$ order over any surface having its edge in the given line, the motion being perpendicular to the surface. Suppose now that the fluid in which this line is placed receives motion of the $(n-1)^{th}$ order, but no higher. In order that this may be possible, it must have a potential, and so will vanish *for this region* when we find the new values of the motion of the n^{th} order for this region. But not so if we add motion of the n^{th} order or higher.

Hence we have the theorem that motion of the n^{th} order is unaffected by motion of any lower order.

We have seen that if within a surface there are closed circuits of the n^{th} order of motion, then without the surface there is motion of the $(n-1)^{th}$ order and below it, the $(n-1)^{th}$ order having a scalar potential.

Hence, if two regions of this nature be placed near each other, the motion of the n^{th} and higher orders will be the same as before, but of the lower orders it will be changed.

In considering the general relations of vectors, I have virtually shown that if the fluid has any motion, then the whole series of motions must exist *somewhere* in space. But we have also seen that if one of the motions, say of the n^{th} order, is confined to any given region, then all the higher motions will also be confined to that region, and the remainder of the fluid will have only the $(n-1)^{th}$ order and below; and that the $(n-1)^{th}$ order outside a given region has a potential satisfying Laplace's equation, but the lower orders do not.

Furthermore, I have also shown how to find the motions of any order outside the given region by integrating throughout the given region.

Now suppose a given region has a distribution of motion of the n^{th} order within it which is discontinuous at the surface.

We have seen how to find the distribution of motion of the n^{th} order without it, in order to satisfy the equation of continuity. We have seen that this distribution has a potential without the region, and that only motion of the n^{th} order and below exists outside the region.

Now let us suppose that the original distribution of the n^{th} order within the region is also such that within that region a potential satisfying Laplace's equation exists, but not the same as that without. Then it is evident that all the motion of the $(n+1)^{th}$ and higher orders exists only at the surface of separation of the two regions, and that both within and without the region only the n^{th} and lower orders of motion exist.

Thus this surface constitutes a surface of discontinuity in the fluid motion of the n^{th} order.

Helmholtz has discussed the case of surfaces of discontinuity when the motion on the two sides was of the zero order, and thus the motion at the surfaces of the first and higher orders.

We have treated this case very fully in the theory of vectors, and have there shown the exact distribution of the motions.

The process deduced from this theory which is most useful in hydrodynamics is the following: —

Let us have a distribution of the $(n+1)^{\text{th}}$ in closed curves so that it satisfies the equation of continuity. Then throughout the rest of space the motion of the $(n+1)^{\text{th}}$ order will be zero, and the n^{th} order will have a cyclic potential $\xi_{n,0}$, the source of the fluid motion being supposed to be the $(n+1)^{\text{th}}$ motion. Select any stream surface which will thus enclose some of the $(n+1)^{\text{th}}$ motion.

Draw the equipotential surfaces for the n^{th} order of motion so as to intersect the given stream surface in lines, and distribute over the surface, in the direction of these lines, motion of the $(n+1)^{\text{th}}$ order, so that

$$M_{n+1}d\nu = M_n$$

at every point of the surface. Then the new distribution of the $(n+1)^{\text{th}}$ motion produces the same effect as the old. Therefore we can take away the old distribution within the surface and replace it by fluid for which the potential of the n^{th} order of motion is constant, and the external effect of the new system is the same as the old.

By putting one stream surface within the other a solid distribution of the $(n+1)^{\text{th}}$ order might be made. If we draw the equipotential surface in the manner that Maxwell has done, we can define the distribution of the $(n+1)^{\text{th}}$ order as follows: —

FIG. 1.

Let the figure represent a system of equipotential surfaces infinitely near together, intersected by two stream surfaces very near together.

Let $d\lambda$ be the distance apart of two equipotential surfaces. Then we can write

$$M_{n+1} d\nu d\lambda = M_n d\lambda.$$

But we also have

$$M_n = -\frac{d\xi_{n,0}}{d\lambda},^*$$

therefore we have

$$M_{n+1} d\nu d\lambda = -\frac{d\xi_{n,0}}{d\lambda} d\lambda.$$

But from the method of drawing the figure the last member is constant, and consequently the first is also. But the first is the total quantity of M_{n+1} within the rectangle, therefore the proper distribution of M_{n+1} is given by putting the same amount into each rectangle. Let us now attempt to get the total *quantity* of M_{n+1}. By *quantity* of M_{n+1} I mean the surface integral of its cross section. Thus, in this case, the quantity of M_{n+1} is

$$\int M_{n+1} d\nu d\lambda = -\int \frac{d\xi_{n,0}}{d\lambda} d\lambda,$$

the integrals to be taken around the section of the surface. The last integral is, in our notation, simply equal to the original *quantity* of $(n+1)^{\text{th}}$ motion. Hence, the quantity of $(n+1)^{\text{th}}$ motion distributed over the surface must be equal to the original quantity. The analogy of this to the case of electric distribution is apparent, the surface integral over the cross section of a vector taking the place of quantity of electricity.

If we apply this to the case of vortex and ordinary motion, we shall have the case of liquid at rest enclosed within a vortex surface, with the motion of the exterior liquid irrotational and expressed by a cyclic potential.

The vortices of the surface are constantly moving forward in the direction of the fluid motion immediately outside the surface.

If we wish a volume distribution of M_{n+1} within the surface, we merely have, as I have shown above, to put surfaces within one another with the above relative distribution, the strength of each surface being arbitrary. We have thus one other condition to fulfil arbitrarily. Let Q_n be a function which is constant for the stream surfaces, or, in other words, the stream function.

Then the solid distribution of M_{n+1} is represented by the formula

$$M_{n+1} = \frac{C}{d\nu dl} \psi(Q_n),$$

where C is a very minute constant, and $\psi(Q_n)$ is a function of Q_n.

* The notation here used is that of the general theory given on p. 236, so that $\xi_{n,0}$ is the same as $-X_n$ used before.

The direction of M_{n-1} is that of the intersection of the surfaces $Q_n =$ constant, and $\xi_{n.0} =$ constant.

Such a distribution of M_{n-1} will produce a potential outside the surface which will be proportional to $\xi_{n.0}$ and equal to it, if C has the proper value. But inside the surface the motion is dependent upon the form of $\psi(Q_n)$.

The distribution of the vortices over all the surfaces need not always be in the same direction, but we may alternate as often as we please without altering the outer distribution of motion.

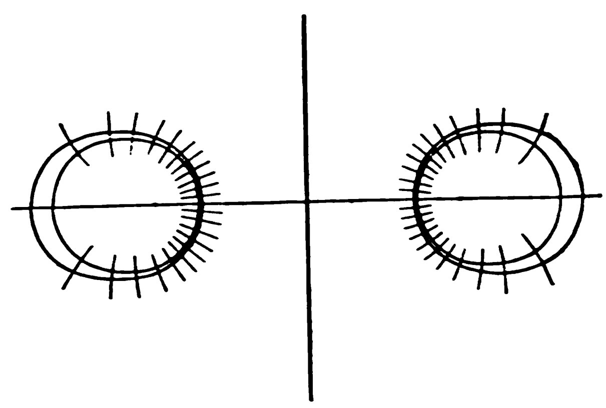

FIG. 2.

As an illustration let us take a circular vortex ring. The equipotential surfaces and lines of flow for an infinitely thin vortex are represented by the preceding diagram, where the closed curves of somewhat circular form are the sections of the surfaces of flow, and the others are the sections of the equipotential surfaces.

The motion of zero order is distributed throughout space according to the cyclic potential $\xi_{n,0}$, and the motion of the first order (vortex motion) is confined to the closed surface.

If we should cause instantaneous vortical forces to act over a surface similar to the given one, the forces being proportional to the vortices in the first case, then the analysis of vector motions shows that these forces produce the same effect as the original distribution of vortical forces for all points outside, but no effect on the points inside the surface. Hence, the system formed by these forces will be the same as the system under discussion. The system thus formed will undergo changes which are investigated in the portion of this paper treating of the equilibrium of fluid motion. We have there seen that the whole system of changes can be treated from the drifting of the vortex filaments, the whole

dynamics of the problem being satisfied by making the product of the vortex-strength of each filament by its cross-section constant as the element moves along. In the present case, the elementary vortex filaments of the vortex surface are constantly drifting from one place to another, and they may thus finally obtain another distribution, in which case the form of the surface and its position may change. In the present case, if the fluid moves through the ring from left to right, and we refer to the upper section, the vortex strength tends to increase on the outer and right-hand side, and decrease on the inner and left-hand side.

When there is any change of distribution, the ring as a whole will tend to revolve around the part which has increased the most, and so the whole ring tends, as a whole, to decrease in diameter and move forward, though the changes of form are not so obvious.

As the ring moves, the weaker part constantly tends to go inside, and the stronger part outside, and so tends to a form of equilibrium, though this should be investigated more thoroughly.

In treating a filament, we must descend to the elements of the filament itself and consider the motion of each secondary element as affected by the others. It is impossible to treat this filament or vortex surface as a whole. The fineness of filaments to which we must descend will depend upon the rate of variation of the vortex strength. Thus, when we are dealing with finite distributions of vortices without *side* discontinuity, the elementary filaments may be large, but when we treat filaments or vortex surfaces, the elementary filaments must be small in proportion.

If we now consider the hollow vortex to be very small in cross section, compared with the radius, so that it is nearly circular, and so does not change its shape much, it is easy to see that the tendency of the elementary vortices to accumulate on the outside will tend to revolve the whole circle into a position farther on in the direction of the fluid motion inside the ring, and if we still further consider that there is a tendency to accumulate in the front part also, we see that the ring will tend to move forward and to contract in radius. If the cross section is very small, the form of the cross section becomes more and more stable. The radius of the ring finally becomes constant when the forward motion is such as to make the difference of velocity between the fluid in the inside of the hollow vortex and the outside, the same for every portion of the cross section of the surface, which is possible for a very small cross section, or for a large one if the shape is correct.

In attempting to construct synthetically a hollow vortex which shall move

forward uniformly through space without change of form, we have simply to find a distribution of elementary vortex rings such that the fluid motion due to them combined with a uniform forward motion shall produce a stream surface along which the motion of the fluid is everywhere uniform as reckoned from a system of axes moving with the surface.

Or, for the sake of ease in calculating, we can suppose the axes at rest and the fluid flowing past them. It would thus be perfectly possible to construct synthetically a hollow vortex ring of finite cross section and finite velocity of translation.

In the case of a very thin, hollow vortex of circular cross section, the velocity can be obtained by finding the motion of the fluid in the interior due to all the vortex filaments of which the ring is formed. Or it is simply one half of the difference of velocity on the outside of the surface nearest and furthest from the axes of symmetry.

The velocity of translation of very thin solid vortex rings can readily be found by calculating the velocity at the core due to the given distribution of vortex filaments constituting the ring. It will thus be found that for a given distribution of vortex filaments within the very thin vortex, the vortex must move perpendicular to the plane of its curvature and inversely as the radius of curvature, to insure stability. Thus, if we curve a circular vortex at any point outward, at first the curved part will move faster than the other part. Then each side of the bend will move outwards, and thus the ring will tend towards a circular form again, though it possibly oscillates around it.

It is evident that whatever changes a hollow vortex may undergo the inner fluid can never mix with the outer, but the vortex surface forms a box through which not a drop of the fluid can escape however the surface may be twisted.

Thomson and others have pronounced all such discontinuous fluid motion to be unstable, and to prove this he supposes a depression to be made in the surface, which he then supposes to increase indefinitely. That some surfaces are thus unstable no one can doubt. But I do not think that all are so. Besides, if we have a vortex surface drifting around in a fluid, the latter having only an irrotational motion, the depression which will form in the surface at any time will have to satisfy certain conditions which may neutralize the previous effect. I have given my reasons for regarding the hollow vortex of uniform strength of surface as the form toward which hollow vortices tend when distorted. But I am not yet prepared to give further results, but will wait to obtain an exact solution of the problem. It is very probable that, when disturbed, oscillations are set up. All the principles necessary for the solution are developed in this paper.

If it is found that the hollow vortex is unstable, it may still be possible to build up a solid stable vortex on the principles here set forth. The condition thus developed for a vortex surface which shall not change its form by the mutual action of its parts is that the strength of the vortex surface shall be constant at every part and the surface a stream surface.

But in order that such a surface may exist, it must be remembered that the components of the vortex motion must satisfy the equation of continuity and the motion of the fluid inside the surface must be acyclic. This last condition is only necessary in order to avoid vortex motion in other parts than the sheet, which, however, may exist if we please. Having constructed such a sheet and calculated the fluid motion from it, it will have a potential everywhere without the sheet satisfying Laplace's equation, without vortex motion. If we construct a hollow vortex ring in this manner, with the condition that the surface be a stream surface, the fluid within the hollow part will have an acyclic motion, but such as not to affect the form of the surface. Thus the motion might be one of translation, and it would be thus possible to have a stable hollow vortex. The only form of surface which can exist without changing its form is thus symmetrical around the axis of motion, and has a cross-section which has not yet been investigated.

In certain cases the changes through which a hollow vortex goes are periodical, and it is a question whether, with a proper motion of translation, they are not always of this nature.

Let there be a distribution $\bar{F}_n$, $\bar{G}_n$, $\bar{H}_n$ inside a given surface. Then F_n, G_n, H_n will have a potential satisfying Laplace's equation at all points where $\bar{F}_n = 0$, $\bar{G}_n = 0$, and $\bar{H}_n = 0$, of the value $\xi_{n,0} = -\chi_n$.

$$\xi_{n,0} = \frac{1}{4\pi}\iiint\left\{\bar{F}_n\frac{d\frac{1}{r}}{dx} + \bar{G}_n\frac{d\frac{1}{r}}{dy} + \bar{H}_n\frac{d\frac{1}{r}}{dz}\right\}dxdydz,$$

or integrating by parts

$$\xi_{n,0} = \frac{1}{4\pi}\iint\frac{1}{r}\{\bar{F}_n l + \bar{G}_n m + \bar{H}_n n\}\,dS - \frac{1}{4\pi}\iiint\frac{1}{r}\left\{\frac{d\bar{F}_n}{dx} + \frac{d\bar{G}_n}{dy} + \frac{d\bar{H}_n}{dz}\right\}dxdydz.$$

If within the surface

$$\bar{F}_n = \frac{d\phi_n}{dx}, \qquad \bar{G}_n = \frac{d\phi_n}{dy}, \qquad \text{and } \bar{H}_n = \frac{d\phi_n}{dz}$$

we have

$$\xi_{n,0} = \frac{1}{4\pi}\iint\frac{\phi_n}{r^2}\cos\theta dS + \phi_n,$$

where θ is the angle between the normal to the surface and r, and the last term

disappears outside the surface where $\bar{F}_n$, $\bar{G}_n$, and $\bar{H}_n$ do not exist. Whence outside the surface

$$\xi'_{n,0} = \frac{1}{4\pi}\iint \frac{\phi_n}{r^2} \cos\theta dS$$

and inside

$$\xi' = \frac{1}{4\pi}\iint \frac{\phi_n}{r^2} \cos\theta dS + \phi_n.$$

The integral is also evidently different outside and inside the surface. We evidently have also

$$\Delta^2\xi'_{n,0} = 0 \text{ outside the surface,}$$

and

$$\Delta^2\xi_{n,0} = 0 \text{ inside the surface.}$$

Hence this is a case similar to the one before it, and the surface has a distribution over it of motion of the $(n+1)^{\text{th}}$ order given by the equations found there; this motion is confined to the surface, and forms closed circuits on it in order to satisfy the equation of continuity.

When motion of the $(n+1)^{\text{th}}$ order is so distributed over the surface that the component of the n^{th} order of motion is zero in the direction of the normal to the surface, then, from what we have before proved, the motions of the $(n+1)^{\text{th}}$ order and higher orders will be confined to the surface, and will not appear in the remainder of the fluid. Thus in the motion resulting from a solid body moving in the fluid, the motion of the zero order is zero at the surface, and so there is no motion of the first or higher orders throughout the fluid, and the motion of the zero order has a potential.

If the discontinuous system $\bar{F}_n$, $\bar{G}_n$, $\bar{H}_n$, whose component is $\bar{M}_n$, are distributed over a surface with the resultant normal to the surface, $d\nu$ being the thickness of the surface, and $\bar{M}_n d\nu$ being constant over the surface, then we have seen that the shell so formed is equivalent to a line of $(n+1)^{\text{th}}$ motion around the edge.

If dS is the cross section of the line, we must have*

$$\bar{M}_n d\nu = M_{n+1} dS.$$

The linear distribution of M_{n+1} is then equivalent to the surface distribution of $\bar{M}_n$. There will be a potential of the n^{th} order throughout space, satisfying Laplace's equation: the motion of the $(n+1)^{\text{th}}$ and higher orders are confined to the boundary line of the surface, and are zero throughout the rest of the space.

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* This is proved in the portion of this paper relating to the energy of the fluid.

So that all space except the line surely contains no motion of the n^{th} and lower orders, the n^{th} only having a potential which satisfies Laplace's equation but is cyclic.

The components of the $(n+1)^{th}$ order are readily obtained by differentiation of the equations

$$F_{n-1} = \frac{M_{n+1} dS}{4\pi} \int \frac{l}{r}\, ds,$$

$$G_{n-1} = \frac{M_{n+1} dS}{4\pi} \int \frac{m}{r}\, ds,$$

$$H_{n-1} = \frac{M_{n+1} dS}{4\pi} \int \frac{n}{r}\, ds,$$

where l, m, and n are evidently the direction cosines of the element of the line ds.

The scalar potential of the n^{th} order is simply proportional to the solid angle subtended by the line at the point, and can be computed by known methods. The components of the n^{th} order can also be simply obtained from the above components of the $(n-1)^{th}$ order by the proper differentiation.

As the motions of the $(n+1)^{th}$ order and higher are confined to the line, the total motion of the fluid can be represented not only by a line integral of the $(n+1)^{th}$ order, but also by a line integral of *any* of the higher orders. The exact form I have not yet investigated.

As an illustration of the methods here given, let us suppose that throughout a sphere of radius R there is a distribution of motion represented by $\bar{F}_n$, while $\bar{G}_n$ and $\bar{H}_n$ are zero. We shall then have

$$O_{n,2} = -\frac{\bar{F}_n}{24} \int\int\int r^3 dxdydz,$$

$$P_{n,2} = 0,$$

$$Q_{n,2} = 0,$$

where the integral is to be taken throughout the interior of the sphere. We thus find

Inside the Sphere. *Outside the Sphere.*

$$O_{n,2} = \frac{\pi \bar{F}_n}{1260}\{35R^6 + 105R^4r^2 + 21R^2r^4 - r^6\}; \quad O'_{n,2} = \frac{\pi \bar{F}_n R^3}{630\, r}\{3R^4 + 42R^2r^2 + 35r^4\};$$

$$P_{n,2} = Q_{n,2} = P'_{n,2} = Q'_{n,2} = 0.$$

Since we have $\Delta^2 r^n = n(n+1)r^{n-2}$, we can write all the following quantities from differentiating these : —

$$O_{n,1} = \frac{\pi \bar{F}_n}{30}\{15R^4 + 10R^2r^2 - r^4\}; \qquad O'_{n,1} = \frac{2\pi R^5 \bar{F}_n}{15r}\{R^2 + 5r^2\};$$

$$P_{n,1} = Q_{n,1} = P'_{n,1} = Q'_{n,1} = 0;$$

$$O_{n,0} = \frac{2\pi \bar{F}_n}{3}\{3R^2 - r^2\}; \qquad O'_{n,0} = \frac{4}{3}\frac{\pi R^3 \bar{F}_n}{r};$$

$$P_{n,0} = Q_{n,0} = P'_{n,0} = Q_{n,0} = 0;$$

$$\Delta^2 O_{n,(-1)} = 4\pi \bar{F}_n; \qquad \Delta^2 O'_{n,(-1)} = 0.$$

From these we have

$$\xi_{n,2} = \frac{\bar{F}_n r}{840}\{35R^4 + 14R^2r^2 - r^4\}\cos\theta; \qquad \xi'_{n,2} = \frac{\bar{F}_n R^5}{120}\{2R^2 + 5r^2\}\cos\theta;$$

$$\xi_{n,1} = \frac{\bar{F}_n r}{30}\{5R^2 - r^2\}\cos\theta; \qquad \xi'_{n,1} = \frac{\bar{F}_n R^5}{30r^2}\{5r^2 - R^2\}\cos\theta;$$

$$\xi_{n,0} = -\frac{\bar{F}_n}{3} r\cos\theta; \qquad \xi'_{n,0} = -\frac{\bar{F}_n R^3}{3}\frac{\cos\theta}{r^2}.$$

Whence we can write the whole system of vectors as follows: —

$$F_{n-4} = \frac{\bar{F}_n}{120}\{15R^4 + 10R^2r^2 - r^4\} - \frac{d\xi_{n,2}}{dx}; \quad F'_{n-4} = \frac{\bar{F}_n R^5}{30r}\{R^2 + 5r^2\} - \frac{d\xi'_{n,2}}{dx};$$

$$G_{n-4} = -\frac{d\xi_{n,2}}{dy}; \qquad G'_{n-4} = -\frac{d\xi'_{n,2}}{dy};$$

$$H_{n-4} = -\frac{d\xi_{n,2}}{dz}; \qquad H'_{n-4} = -\frac{d\xi'_{n,2}}{dz}.$$

$$F_{n-3} = 0; \qquad F'_{n-3} = 0;$$

$$G_{n-3} = -\frac{\bar{F}_n}{30}\{5R^2 - r^2\}z; \qquad G'_{n-3} = -\frac{\bar{F}_n R^5}{30r}\left\{\frac{R^2}{r^2} + 5\right\}z;$$

$$H_{n-3} = \frac{\bar{F}_n}{30}\{5R^2 - r^2\}y; \qquad H'_{n-3} = \frac{\bar{F}_n R^5}{30r}\left\{\frac{R^2}{r^2} + 5\right\}y.$$

$$F_{n-2} = \frac{\bar{F}_n}{6}\{3R^2 - r^2\} - \frac{d\xi_{n,1}}{dx}; \qquad F'_{n-2} = \frac{\bar{F}_n R^3}{3r} - \frac{d\xi'_{n,1}}{dx};$$

$$G_{n-2} = -\frac{d\xi_{n,1}}{dy}; \qquad G'_{n-2} = -\frac{d\xi'_{n,1}}{dy};$$

$$H_{n-2} = -\frac{d\xi_{n,1}}{dz}; \qquad H'_{n-2} = -\frac{d\xi'_{n,1}}{dz}.$$

$$F_{n-1} = 0; \qquad F'_{n-1} = 0;$$

$$G_{n-1} = \frac{\bar{F}_n}{3} z; \qquad G'_{n-1} = \frac{\bar{F}_n R^3}{3} \frac{z}{r^3};$$

$$H_{n-1} = -\frac{\bar{F}_n}{3} y; \qquad H'_{n-1} = -\frac{\bar{F}_n R^3}{3} \frac{y}{r^3}.$$

$$F_n = \bar{F}_n - \frac{d\xi_{n.0}}{dx} = \frac{2}{3} \bar{F}_n; \qquad F'_n = -\frac{d\xi_{n.0}}{dx} = \frac{\bar{F}_n R^3}{3} \frac{r^2 - 3x^2}{r^5};$$

$$G_n = -\frac{d\xi_{n.0}}{dy} = 0; \qquad G'_n = -\frac{d\xi_{n.0}}{dy} = -\frac{\bar{F}_n R^3}{3} \frac{3xy}{r^5};$$

$$H_n = -\frac{d\xi_{n.0}}{dz} = 0; \qquad H'_n = -\frac{d\xi_{n.0}}{dz} = -\frac{F_n R^3}{3} \frac{3xz}{r^5}.$$

The orders higher than these are only distributed over the surface, the next higher order being

$$F_{n+1}\, d\nu = 0,$$

$$G_{n+1}\, d\nu = -\frac{1}{3} \bar{F}_n \frac{z}{R},$$

$$H_{n+1}\, d\nu = \frac{1}{3} \bar{F}_n \frac{y}{R}.$$

The next higher order is distributed, as it were, on each side of the surface whose thickness is $d\nu$.

By giving to n different values up to $n = 4$, we can get various distributions of fluid motion, all of which satisfy the equation of continuity and are thus possible forms of motion. When $n = 4$, the motion of the fluid can be considered as due to a uniform distribution of motion of the fourth order, $\bar{F}_4$, throughout the sphere, the motions of the lower orders being distributed as given by the equations. Or the motion can be considered as due to a distribution of F_{n+1}, G_{n+1}, H_{n+1} over the surface of the sphere. When $n = 0$, the case is that of a sphere of liquid proceeding forward uniformly in the direction of the axis of x.

If we replace the sphere by a solid sphere, the motion of the fluid outside will remain the same as before.

The changes which a sphere moving in the manner described will undergo can be calculated from the drifting of the vortex sheet backwards, as shown further on.

The sphere will thus tend to flatten in the direction of the axis of x and broaden out in the other direction so as to form a figure somewhat similar to a prolate ellipsoid.

It is to be noted that the whole system of motions which I have given above have only required *one* integration, and in the general case would only require *three* for the complete determination of all the vectors up to the $2s$ order.

As another illustration, take a distribution of $\bar{F}_n$ along the axis of x, and make $s = 2$, as before. Let the area of the section of the small tube along the axis be a, and in the integration reject the part which becomes infinite. We thus find

$$O_{n,2} = \frac{\bar{F}_n a}{64} q^4 \log q.$$

where

$$q = \sqrt{y^2 + z^2},$$

whence

$$\xi_{n,2} = 0,$$

and

$$F_{n-4} = \frac{1}{4\pi} \Delta^2 O_{n,2} = \frac{\bar{F}_n a}{8} (q^2 + 2q^2 \log q),$$

$$G_{n-4} = 0,$$

$$H_{n-4} = 0;$$

$$F_{n-3} = 0,$$

$$G_{n-3} = \frac{\bar{F}_n a}{2} (1 + \log q) z,$$

$$H_{n-3} = -\frac{\bar{F}_n a}{2} (1 + \log q) y;$$

$$F_{n-2} = -\frac{\bar{F}_n a}{2} \{3 + 2 \log q\},$$

$$G_{n-2} = 0,$$

$$H_{n-2} = 0;$$

$$F_{n-1} = 0,$$

$$G_{n-1} = -\bar{F}_n a \frac{z}{q^2},$$

$$H_{n-1} = +\bar{F}_n a \frac{y}{q^2}.$$

The calculation of F_n, G_n, H_n makes them zero for all points of space except the axis of x, just as it should.

But we can also consider them as distributed over any surface enclosing the axis of x.

Taking a stream surface for F_{n-1}, G_{n-1}, H_{n-1} which will be any surface of revolution around the axis, the proper distribution of F_n, G_n, H_n over the surface will cause the above values outside the surface, and a zero value of F_{n-1}, G_{n-1}, H_{n-1} inside.

Thus, take a circular cylinder of radius b. We must distribute $\bar{F}_n a$ over the surface of thickness $d\nu$. Therefore we have

$$2\pi b \bar{F}'_n d\nu = \bar{F}_n a,$$

$$\bar{F}'_n d\nu = \frac{\bar{F}_n a}{2\pi b}.$$

Beside this distribution of $\bar{F}_n$ we can distribute it uniformly in the interior of the circular cylinder, or in any other way suggested by the equations. In the case of uniform distribution throughout the cylinder we can replace it by a surface distribution of F_{n+1}, G_{n+1}, H_{n+1} and so on ad infinitum, the exterior distribution of velocities being the same, but the interior being different.

The Action of Forces on Fluids.

Let a system of forces whose components at the point x, y, z are $\bar{X}$, $\bar{Y}$, $\bar{Z}$ act on a fluid, and let p be the pressure of the fluid at the same point. The hydrodynamical equations of Euler are then,

$$\frac{1}{\rho}\frac{dp}{dx} = \bar{X}_0 - \frac{dF_0}{dt} - F_0\frac{dF_0}{dx} - G_0\frac{dF_0}{dy} - H_0\frac{dF_0}{dz},$$

$$\frac{1}{\rho}\frac{dp}{dy} = \bar{Y}_0 - \frac{dG_0}{dt} - F_0\frac{dG_0}{dx} - G_0\frac{dG_0}{dy} - H_0\frac{dG_0}{dz},$$

$$\frac{1}{\rho}\frac{dp}{dz} = \bar{Z}_0 - \frac{dH_0}{dt} - F_0\frac{dH_0}{dx} - G_0\frac{dH_0}{dy} - H_0\frac{dH_0}{dz}.$$

It is usual in treatises on hydrodynamics to consider cases where $\bar{X}_0$, $\bar{Y}_0$, $\bar{Z}_0$ have a potential, and it is there stated that vortex motion cannot be produced by such forces. But if we consider that forces which have a scalar potential are

such as are produced by direct attraction to or repulsion from points distributed throughout the fluid, we see that conservative forces acting in an unlimited medium can never produce *any motion whatever*, but only influence the pressure.

Thus the equations as they stand contain much that is superfluous, and the motion will be the same in every respect if we differentiate in such a manner as to eliminate all portions of X, Y, Z which depend on a scalar potential.

Let us write

$$W = \frac{p}{S} + \tfrac{1}{2} M_0^2 ;$$

then the equations can be put in the form*

$$\frac{dW}{dx} - \bar{X}_0 + \frac{dF_0}{dt} = G_0 H_1 - H_0 G_1,$$

$$\frac{dW}{dy} - \bar{Y}_0 + \frac{dG_0}{dt} = H_0 F_1 - F_0 H_1,$$

$$\frac{dW}{dz} - \bar{Z}_0 + \frac{dH_0}{dt} = F_0 G_1 - G_0 F_1.$$

Let us now write

$$X_1 = \frac{d\bar{Z}_0}{dy} - \frac{d\bar{Y}_0}{dz},$$

$$Y_1 = \frac{d\bar{X}_0}{dz} - \frac{d\bar{Z}_0}{dx},$$

$$Z_1 = \frac{d\bar{Y}_0}{dx} - \frac{d\bar{X}_0}{dy},$$

and also

$$X_2 = \frac{dZ_1}{dy} - \frac{dY_1}{dz} = -\Delta^2 \bar{X}_0 + \frac{d}{dx}\left(\frac{d\bar{X}_0}{dx} + \frac{d\bar{Y}_0}{dy} + \frac{d\bar{Z}_0}{dz}\right),$$

$$Y_2 = \frac{dX_1}{dz} - \frac{dZ_1}{dx} = -\Delta^2 \bar{Y}_0 + \frac{d}{dy}\left(\frac{d\bar{X}_0}{dx} + \frac{d\bar{Y}_0}{dy} + \frac{d\bar{Z}_0}{dz}\right),$$

$$Z_2 = \frac{dY_1}{dx} - \frac{dX_1}{dy} = -\Delta^2 \bar{Z}_0 + \frac{d}{dz}\left(\frac{d\bar{X}_0}{dx} + \frac{d\bar{Y}_0}{dy} + \frac{d\bar{Z}_0}{dz}\right),$$

etc. etc.

Now I have shown in the theory of vector quantities that every system of discontinuous vectors can be replaced by another system which satisfies the equation of continuity, provided we can show a physical reason for the vectors

* Given in a more restricted form by Lamb, in his "Treatise on the Motion of Fluids," p. 241.

satisfying that equation. In the case under consideration, the pressure conducts the force applied at one point to another, and the whole system of forces so applied to the fluid must satisfy the equation of continuity. Hence for the system $\bar{X}_n$, $\bar{Y}_n$, $\bar{Z}_n$, can be substituted the system

$$X_n = \bar{X}_n + \frac{d\chi_n}{dx},$$

$$Y_n = \bar{Y}_n + \frac{d\chi_n}{dy},$$

$$Z_n = \bar{Z}_n + \frac{d\chi_n}{dz},$$

where

$$\chi_n = -\frac{1}{4\pi}\iiint \frac{1}{r}\left(\frac{d\bar{X}_n}{dx} + \frac{d\bar{Y}_n}{dy} + \frac{d\bar{Z}_n}{dz}\right) dxdydz.$$

When this substitution for the zero order is made in the original equations, the only portion of the pressure that will remain will be that which arises from the motion of the fluid and not from the applied forces. In this case we can simply write

$$X_2 = -\Delta^2 X_0; \quad Y_2 = -\Delta^2 Y_0; \quad Z_2 = -\Delta^2 Z_0.$$

Differentiating the third of our equations with respect to y and the second with respect to z, and subtracting, we can write the first of the following series of equations; the other two can be written from symmetry:—

$$X_1 = \frac{\delta F_1}{\delta t} - F_1\frac{dF_0}{dx} - G_1\frac{dF_0}{dy} - H_1\frac{dF_0}{dz},$$

$$Y_1 = \frac{\delta G_1}{\delta t} - F_1\frac{dG_0}{dx} - G_1\frac{dG_0}{dy} - H_1\frac{dG_0}{dz},$$

$$Z_1 = \frac{\delta H_1}{\delta t} - F_1\frac{dH_0}{dx} - G_1\frac{dH_0}{dy} - H_1\frac{dH_0}{dz},$$

where the symbol δ refers to the *moving* element and has the well-known value

$$\frac{\delta}{\delta t} = \frac{d}{dt} + F_0\frac{d}{dx} + G_0\frac{d}{dy} + H_0\frac{d}{dz}.$$

Performing the same operation on these, we have the equations

$$X_2 = \frac{\delta F_2}{\delta t} - F_2\frac{dF_0}{dx} - G_2\frac{dF_0}{dy} - H_2\frac{dF_0}{dz} + 2\left\{\begin{array}{l} \frac{dF_0}{dy}\frac{dF_1}{dz} - \frac{dF_1}{dy}\frac{dF_0}{dz} \\ + \frac{dG_0}{dy}\frac{dG_1}{dz} - \frac{dG_1}{dy}\frac{dG_0}{dz} \\ + \frac{dH_0}{dy}\frac{dH_1}{dz} - \frac{dH_1}{dy}\frac{dH_0}{dz} \end{array}\right\},$$

$$Y_2 = \frac{\delta G_2}{\delta t} - F_2\frac{dG_0}{dx} - G_2\frac{dG_0}{dy} - H_2\frac{dG_0}{dz} + 2\left\{\begin{array}{l} \frac{dF_0}{dz}\frac{dF_1}{dx} - \frac{dF_1}{dz}\frac{dF_0}{dx} \\ + \frac{dG_0}{dz}\frac{dG_1}{dx} - \frac{dG_1}{dz}\frac{dG_0}{dx} \\ + \frac{dH_0}{dz}\frac{dH_1}{dx} - \frac{dH_1}{dz}\frac{dH_0}{dx} \end{array}\right\},$$

$$Z_2 = \frac{\delta H}{\delta t} - F_2\frac{dH_0}{dx} - G_2\frac{dH_0}{dy} - H_2\frac{dH_0}{dz} + 2\left\{\begin{array}{l} \frac{dF_0}{dx}\frac{dF_1}{dy} - \frac{dF_1}{dx}\frac{dF_0}{dy} \\ + \frac{dG_0}{dx}\frac{dG_1}{dy} - \frac{dG_1}{dx}\frac{dG_0}{dy} \\ + \frac{dH_0}{dx}\frac{dH_1}{dy} - \frac{dH_1}{dx}\frac{dH_0}{dy} \end{array}\right\}.$$

When a fluid in motion is left to itself, the problem of the changes which it undergoes has never been satisfactorily solved, even in the case of a single vortex ring left in space by itself, and the principles to guide one in the solution of the problem have not been very satisfactorily given.

When the forces acting on the fluid are zero, we have

$$\frac{\delta F_1}{\delta t} = F_1\frac{dF_0}{dx} + G_1\frac{dF_0}{dy} + H_1\frac{dF_0}{dz},$$

$$\frac{\delta G_1}{\delta t} = F_1\frac{dG_0}{dx} + G_1\frac{dG_0}{dy} + H_1\frac{dG_0}{dz},$$

$$\frac{\delta H_1}{\delta t} = F_1\frac{dH_0}{dx} + G_1\frac{dH_0}{dy} + H_1\frac{dH_0}{dz}.$$

These equations contain the whole dynamics of the subject, and simply show that the product of the strength of the first order of motion by its cross section is constant. This principle then, applied in the proper way, contains the whole of the dynamics of a perfect fluid.

The process indicated by the equations is as follows: Having given certain values of the components of the velocities, F_0, G_0, H_0, which satisfy the equation of continuity, we calculate from these the distribution of the first order of motion.

The variation of this first order of motion in each element as it drifts along is given by the above equations, or the variation of motion in any fixed element is given by these modified equations.

The variation of the fluid motion can then be calculated from these. Again applying the method, we could get a still further change, and if we continued this step by step process indefinitely we might trace out the whole fluid motion. But we might originally obtain F_0, G_0, H_0 as functions of x, y, z, and t, so that for $t =$ constant, they should satisfy the equation of continuity, and for t variable, the above equations.

The result so found would give all the changes which a given system would undergo which was started at a given time in any one of the configurations. *But it is to be particularly noted that in using these equations we must always descend to such small elements of the fluid that discontinuity is avoided.*

Thus we should never treat a single vortex filament by itself, but should descend to the still smaller filaments of which the vortex filament is composed, and apply our equations to these.

In this manner the problem of the motion of a single vortex filament becomes perfectly determinate instead of indeterminate as before. And I believe that a recognition of this subject will lead to extremely important results in this subject. It is, then, in the study of these differential equations that the final solution is to be obtained.

We can always tell from the equations in which direction the system tends to change, and can thus base a theory of the stability of fluid motion on them. But I have not yet attempted to find any solutions. It is to be noted that the equations are satisfied at all points of space for which the first order of motion is zero, and so we can always confine our attention to points where it exists.

But we are able to regard the matter from another point of view. When no external forces act on the fluid, the equations of Euler become

$$\frac{1}{\rho}\frac{dp}{dx} = -\frac{\delta F_0}{\delta t},$$

$$\frac{1}{\rho}\frac{dp}{dy} = -\frac{\delta G_0}{dt},$$

$$\frac{1}{\rho}\frac{dp}{dz} = -\frac{\delta H_0}{dt},$$

where

$$-\frac{\delta F_0}{\delta t}, \qquad -\frac{\delta G_0}{\delta t}, \qquad -\frac{\delta H_0}{\delta t},$$

are the forces of acceleration of the fluid. Hence we have

$$\frac{d}{dy}\frac{\delta H_0}{\delta t} - \frac{d}{dz}\frac{\delta G_0}{\delta t} = 0,$$

$$\frac{d}{dz}\frac{\delta F_0}{\delta t} - \frac{d}{dx}\frac{\delta H_0}{\delta t} = 0,$$

$$\frac{d}{dx}\frac{\delta G_0}{\delta t} - \frac{d}{dy}\frac{\delta F_0}{\delta t} = 0.$$

These equations simply express the fact that in a fluid not acted upon by external forces the forces of acceleration are acyclic. One of the most interesting cases is that of a surface of discontinuity. In this case the equations assume the form

$$\left(\frac{\delta H_0'}{\delta t} - \frac{\delta H_0}{\delta t}\right)\frac{d\nu}{dy} - \left(\frac{\delta G_0'}{\delta t} - \frac{\delta G_0}{\delta t}\right)\frac{d\nu}{dz} = 0,$$

$$\left(\frac{\delta F_0'}{\delta t} - \frac{\delta F_0}{\delta t}\right)\frac{d\nu}{dz} - \left(\frac{\delta H_0'}{\delta t} - \frac{\delta H_0}{\delta t}\right)\frac{d\nu}{dx} = 0,$$

$$\left(\frac{\delta G_0'}{\delta t} - \frac{\delta G_0}{\delta t}\right)\frac{d\nu}{dx} - \left(\frac{\delta F_0'}{\delta t} - \frac{\delta F_0}{\delta t}\right)\frac{d\nu}{dy} = 0.$$

To get the meaning of these equations let us transform them until the normal to the surface is in the direction of the axis of X. We then have

$$\frac{\delta H_0'}{\delta t} - \frac{\delta H_0}{\delta t} = 0,$$

$$\frac{\delta G_0'}{\delta t} - \frac{\delta G_0}{\delta t} = 0.$$

And we also have for the continuity of the fluid

$$F_0' - F_0 = 0.$$

At all points of the fluid where there is no discontinuity, but the fluid velocities are obtained from a potential, these equations are satisfied. It is only where vortices exist that there is a possibility of the equations not holding.

The equations show that there must be no discontinuity in the forces of acceleration at the surface and parallel to it, though there may be in the direction normal to the surface.

If we then have a surface of discontinuity of this nature, we must then add to the forces of acceleration in the interior of the surface others so as to make the system continuous at the surface in all directions except that perpendicular to the surface. This is the same thing as saying that forces must exist throughout the interior of the surface tending to change the configuration.

These forces must evidently be acyclic within the surface, and have the proper value at the surface, and so are perfectly determined.

As an illustration of these methods of finding what dynamical changes will take place in a fluid, let us take the case of a single vortex filament along the axis of X. The fluid velocity will be inversely as the distance from the axis, or

$$F_0 = 0,$$
$$G_0 = F_1 a \frac{z}{q^2},$$
$$H_0 = -F_1 a \frac{y}{q^2},$$

where $q = \sqrt{y^2 + z^2}$, and a is the sectional area of the filament.

Let us take a stream surface bounded by $x = b$ and $x = 0$ and $q = c$ and $q = e$, and distribute $F_1 a$ over it according to our equations. Then, on taking away the original distribution of vortex motion, the fluid will move within the surface the same as before, but without will be at rest. We immediately see that the system is not in equilibrium, for the centrifugal force of the moving liquid remains unbalanced by the fluid pressure. We readily see that the whole ring will expand indefinitely.

The second method expresses the fact as follows: For the surface $q = c$ and $q = e$ the vortex strength has of itself no tendency to vary.

But on the plane surfaces we have

$$\frac{\delta F_1}{\delta t} = 0,$$
$$\frac{\delta G_1}{\delta t} = -F_1 a \left\{ G_1 \frac{2yz}{q^4} + H_1 \frac{z^2 - y^2}{q^4} \right\},$$
$$\frac{\delta H_1}{\delta t} = -F_1 a \left\{ G_1 \frac{z^2 - y^2}{q^4} - H_1 \frac{2yz}{q^4} \right\}.$$

Now the distribution of G_1 and H_1 is in the radial direction, and so these equations show that the direction of the vortex elements tends to change to one in a direction perpendicular to q. Thus we arrive at the same direction of change as before.

By the other method of looking upon the problem, we see that forces of acceleration must exist in the stationary fluid, and so the stationary portion will tend to move. The direction is readily determined.

We are now prepared to state what the conditions are that there shall be no tendency to change from its present configuration.

For a surface of discontinuity the condition is simply that the surface be a stream surface and that the strength of the surface, $Md\nu$, be constant, or that the fluid velocity on the two sides of the sheet be everywhere equal and opposite in direction.

This investigation has led us to look upon the subject of vortex motion from a broader point of view than before. For we have seen that if we reckon up the amount of this motion by the number of fluid elements multiplied by the strength of the motion in them, then the motion so reckoned is being constantly created and destroyed by the other fluid motions. But if we reckon the quantity of vortex motion by its surface integral taken across its cross section, or by its *circulation*, then the statement that it is indestructible by any motion lower than itself is perfectly correct. And this latter definition of the *quantity* of any vector is so important in the theory of the replacement of one system of vectors by another, that I propose that the term obtain general use.

The equilibrium of fluid motion can evidently be either stable, unstable, or neutral. Thus an infinitely long cylinder of fluid revolving around the axis in a medium at rest would evidently be in unstable equilibrium. Thomson has given the criteria for such cases in terms of the energy of the system, and I am not yet prepared to discuss the subject further.

But his conclusion as to the instability of all cases of discontinuous fluid motion I am not willing to admit. The case of the hollow vortex, it seems to me, shows that there can be stable forms. For the changes which the system undergoes are toward stability, seeing that the vortex distribution of the sheet tends to become uniform, which is a case of equilibrium.

Again, the sheet can never be broken and the inside fluid mix with the outside. Hence I am of the opinion that a hollow vortex ring is stable, and always tends to the form where the vortex strength of the sheet is uniform over the surface. But the subject should be carefully examined before a final decision can be reached.

To determine the pressure in fluid motion we can use the equations

$$\frac{1}{\rho}\frac{dp}{dx} = -\frac{\delta F_0}{\delta t},$$

$$\frac{1}{\rho}\frac{dp}{dy} = -\frac{\delta G_0}{\delta t},$$

$$\frac{1}{\rho}\frac{dp}{dz} = -\frac{\delta H_0}{\delta t},$$

whence we have

$$\Delta^2 p = -\rho\left\{\frac{d}{dx}\frac{\delta F_0}{\delta t} + \frac{d}{dy}\frac{\delta G_0}{\delta t} + \frac{d}{dz}\frac{\delta H_0}{\delta t}\right\},$$

whence

$$v = \frac{\rho}{4\pi}\iiint\frac{1}{r}\left\{\frac{d}{dx}\frac{\delta F_0}{\delta t} + \frac{d}{dy}\frac{\delta G_0}{\delta t} + \frac{d}{dz}\frac{\delta H_0}{\delta t}\right\}dxdydz.$$

This can be put in the form

$$p = -\frac{\rho}{2\pi}\iiint\left\{\frac{dF_0}{dy}\frac{dG_0}{dx} - \frac{dF_0}{dx}\frac{dG_0}{dy} + \frac{dF_0}{dz}\frac{dH_0}{dx} - \frac{dF_0}{dx}\frac{dH_0}{dz} + \frac{dG_0}{dz}\frac{dH_0}{dy} - \frac{dG_0}{dy}\frac{dH_0}{dz}\right\}dxdydz.$$

We may also write a very interesting form from integration by parts

$$p = -\frac{\rho}{4\pi}\iiint\left\{\frac{\delta F_0}{\delta t}\frac{d\frac{1}{r}}{dx} + \frac{\delta G_0}{\delta t}\frac{d\frac{1}{r}}{dy} + \frac{\delta H_0}{\delta t}\frac{d\frac{1}{r}}{dz}\right\}dxdydz.$$

If the forces of acceleration have a potential throughout a given region, these reduce to surface integrals. If the forces of acceleration form closed circuits, no fluid pressure exists; that is, if the forces of acceleration satisfy the equation of continuity.

In this case, since we have

$$\frac{d}{dx}\frac{dF_0}{dt} + \frac{d}{dy}\frac{dG_0}{dt} + \frac{d}{dz}\frac{dH_0}{dt} = 0,$$

if we write

$$R_0 = F_0\frac{dF_0}{dx} + G_0\frac{dF_0}{dy} + H\frac{dF_0}{dz},$$

$$S_0 = F_0\frac{dG_0}{dx} + G_0\frac{dG_0}{dy} + H_0\frac{dG_0}{dz},$$

$$T_0 = F_0\frac{dH_0}{dx} + G_0\frac{dH_0}{dy} + H_0\frac{dH_0}{az},$$

we shall then have

$$v = \frac{\rho}{4\pi}\iiint\frac{1}{r}\left\{\frac{dR}{dx} + \frac{dS}{dy} + \frac{dT}{dz}\right\}dxdydz,$$

whence

$$p = -\frac{\rho}{4\pi}\iiint\left\{L_0\frac{d\frac{1}{r}}{dx} + M_0\frac{d\frac{1}{r}}{dy} + N_0\frac{d\frac{1}{r}}{dz}\right\}dxdydz.$$

If M_1 is the resultant vortex motion and M_0 the resultant ordinary motion, and we draw ν in the direction of M_0, we can write the equation in the form

$$p = \frac{\rho}{2\pi}\iiint\frac{1}{r}\left\{\left(\frac{dM_0}{d\nu}\right)^2 - M_1^2\right\}dxdydz,$$

or, as we prefer to write it,

$$\Delta^2 p = 2\rho\left\{M_1^2 - \left(\frac{dM_0}{d\nu}\right)^2\right\}.$$

If points attracting as the square of the distance be distributed through space with a density

$$2\left\{\left(\frac{dM_0}{d\nu}\right)^2 - M_1^2\right\}$$

at every point, then the pressure would be the same as that due to the motion of the fluid.

In this sense the above expression may be considered as the *source* of the fluid pressure.

Let us now take the equations of page 260, and write

$$I = G_0 H_1 - H_0 G_1,$$
$$J = H_0 F_1 - F_0 H_1,$$
$$K = F_0 G_1 - G_0 F_1,$$

whence we have, calling C a constant,

$$p = C - \frac{\rho}{2} M_0^2 - \frac{\rho}{4\pi} \iiint \frac{1}{r} \left\{ \frac{dI}{dx} + \frac{dJ}{dy} + \frac{dK}{dz} \right\} dxdydz,$$

which can be put in the form

$$p = C - \frac{\rho}{2} M_0^2 - \frac{\rho}{4\pi} \iiint \frac{1}{r} \{ M_1^2 - F_0 F_2 - G\ G_2 - H_0 H_2 \} dxdydz.$$

We can also, from integration by parts, as the surface integral vanishes when the integral is taken throughout space, put

$$p = C - \frac{\rho}{2} M_0^2 + \frac{\rho}{4\pi} \iiint \left\{ I \frac{d\frac{1}{r}}{dx} + J \frac{d\frac{1}{r}}{dy} + K \frac{d\frac{1}{r}}{dz} \right\} dxdydz.$$

This expression can be written in the form

$$p = C - \frac{\rho}{2} M_0^2 + \phi,$$

where ϕ satisfies Laplace's equation at all points where no vortices exist. In the expression as ordinarily given, ϕ is the potential of the applied forces, but here it is given in terms of the vortex motion.

Some of these expressions for the pressure are similar to those obtained by Mr. Craig, and published in the Journal of the Franklin Institute.

On certain Possible Cases of Steady Motion in a Viscous Fluid.

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THE following paper contains, first, some general principles governing steady motion in viscous fluids; second, the detailed working out of two cases, (*i*) a sphere moving with constant velocity in the direction of the axis of x, (*ii*) an ellipsoid moving uniformly in the same direction. The results obtained are certain to hold for slow motions, though they have been obtained without that assumption, but it is not proved that the prescribed conditions will exist for rapid motions. If it can be shown that a velocity can be chosen for the moving body, so that the quantity

$$\Delta^2 u \,.\, dx + \Delta^2 v \,.\, dy + \Delta^2 w \,.\, dz$$

shall be an exact differential, then the solution below given will hold for that case, and for that case only.

Part of what immediately follows I have already given in another place, but it is repeated here for convenience.

The expressions for the fluid pressure in different cases are given in the "Journal of the Franklin Institute," for October, 1880. The values found for the velocities of a fluid particle when a sphere moves in any direction in the fluid are given in the "Philosophical Magazine" for November, 1880. A slight error exists in these values as there given, which is corrected here.

Denote by u, v, w the component velocities of a fluid particle in the direction of the axes x, y, z; p, the density of the fluid at the point x, y, z; ρ, the constant density, and μ, the coefficient of viscosity; the kinematic coefficient of viscosity or the ratio of μ to ρ will be denoted by k, i. e. $\frac{\mu}{\rho} = k$.

The equations of motion of an incompressible viscous fluid are now

$$\frac{du}{dt} = X - \frac{1}{\rho}\frac{dp}{dx} + k\Delta^2 u,$$
$$\frac{dv}{dt} = Y - \frac{1}{\rho}\frac{dp}{dy} + k\Delta^2 v, \qquad (1)$$
$$\frac{dw}{dt} = Z - \frac{1}{\rho}\frac{dp}{dz} + k\Delta^2 w,$$

X, Y, Z being external forces. In what follows we will suppose the forces X, Y, Z to possess a potential. If the motion of the fluid is caused by a body which has been projected in it, and is acted upon by forces due to a potential, the potential must be of the form

$$Ax + By + Cz,$$

for the motion to be steady relatively to the body, as a constant resistance has then to be overcome. If the body is at rest and the liquid streaming past it, the potential must contain a term of the form

$$Ax + By + Cz$$

at infinity, to keep up the steady motion; otherwise the motion would die away and the liquid come to rest from the presence of factors of the form e^{-pt}.*

Denoting by ξ, η, ζ the component angular velocities of the fluid particle at the point x, y, z, we have

$$\xi = \frac{1}{2}\left(\frac{dw}{dy} - \frac{dv}{dz}\right),$$
$$\eta = \frac{1}{2}\left(\frac{du}{dz} - \frac{dw}{dx}\right), \qquad (2)$$
$$\zeta = \frac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy}\right).$$

From these follow readily the known relations

$$\Delta^2 u = 2\left(\frac{d\eta}{dz} - \frac{d\zeta}{dy}\right),$$
$$\Delta^2 v = 2\left(\frac{d\zeta}{dx} - \frac{d\xi}{dz}\right), \qquad (3)$$
$$\Delta^2 w = 2\left(\frac{d\xi}{dy} - \frac{d\eta}{dx}\right).$$

* I am indebted to Mr. Greenhill of Emanuel College, Cambridge, for the above remarks, and also for many other most valuable suggestions.

Denote by Ω the resultant angular velocity; then

$$\Omega^2 = \xi^2 + \eta^2 + \zeta^2; \tag{4}$$

also write

$$2\,q = u^2 + v^2 + w^2. \tag{5}$$

The internal friction involves a certain dissipation of energy; the function expressing the rate of dissipation per unit volume has been called by Lord Rayleigh the "dissipation-function;" denoting this by E, we have

$$\begin{aligned} E = 2\,\mu\Big\{\left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dy}\right)^2 + \left(\frac{dw}{dz}\right)^2 + \frac{1}{2}\left(\frac{dw}{dy} + \frac{dv}{dz}\right)^2 \\ + \frac{1}{2}\left(\frac{du}{dz} + \frac{dw}{dx}\right)^2 + \frac{1}{2}\left(\frac{dv}{dx} + \frac{du}{dy}\right)^2\Big\}. \end{aligned} \tag{6}$$

This can be given in a different form by obtaining the expression for $\Delta^2 q$; this is readily found to be

$$\begin{aligned} \Delta^2 q = u\Delta^2 u + v\Delta^2 v + w\Delta^2 w \\ + \left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2 \\ + \left(\frac{dv}{dx}\right)^2 + \left(\frac{dv}{dy}\right)^2 + \left(\frac{dv}{dz}\right)^2 \\ + \left(\frac{dw}{dx}\right)^2 + \left(\frac{dw}{dy}\right)^2 + \left(\frac{dw}{dz}\right)^2. \end{aligned} \tag{7}$$

Add $2\,\Omega^2$ to $\frac{E}{2\,\mu}$, and we will eliminate terms of the form

$$\frac{dw}{dy}\cdot\frac{dv}{dz}, \text{ etc.}$$

Then compare the resulting form of equation (6) with equation (7), and we have at once

$$E = 2\,\mu\{\Delta^2 q - (u\Delta^2 u + v\Delta^2 v + w\Delta^2 w) - 2\,\Omega^2\} \tag{8}$$

or

$$u\Delta^2 u + v\Delta^2 v + w\Delta^2 w = \Delta^2 q - \frac{1}{2\,\mu}E - 2\,\Omega^2. \tag{9}$$

In equations (1) the quantities on the left-hand sides may be replaced by

$$\frac{du}{dt} + u\,\frac{du}{dx} + v\,\frac{dv}{dx} + w\,\frac{dw}{dx} + 2\,(w\eta - v\zeta), \text{ etc.,}$$

or by

$$\frac{du}{dt} + \frac{dq}{dx} + 2\,(w\eta - v\zeta), \text{ etc.}$$

On making these changes, the equations of motion become

$$\left.\begin{aligned}
\frac{du}{dt} + \frac{d(V+q)}{dx} + \frac{1}{\rho}\frac{dp}{dx} + 2(w\eta - v\zeta) &= k\Delta^2 u,\\
\frac{dv}{dt} + \frac{d(V+q)}{dy} + \frac{1}{\rho}\frac{dp}{dy} + 2(u\zeta - w\xi) &= k\Delta^2 v,\\
\frac{dw}{dt} + \frac{d(V+q)}{dz} + \frac{1}{\rho}\frac{dp}{dz} + 2(v\xi - u\eta) &= k\Delta^2 w.
\end{aligned}\right\} \tag{10}$$

To these is to be added the equation of continuity,

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0. \tag{11}$$

Write

$$P = V + q + \int \frac{dp}{\rho};$$

then, introducing the conditions for steady motion, (10) become

$$\left.\begin{aligned}
\frac{dP}{dx} + 2(w\eta - v\zeta) &= k\Delta^2 u,\\
\frac{dP}{dy} + 2(u\zeta - w\xi) &= k\Delta^2 v,\\
\frac{dP}{dz} + 2(v\xi - u\eta) &= k\Delta^2 w.
\end{aligned}\right\} \tag{12}$$

If we assume that the quantities $\Delta^2 u$, $\Delta^2 v$, $\Delta^2 w$ are the first differential coefficients with respect to x, y, z of a function Q, these equations become

$$\left.\begin{aligned}
\frac{d(P-kQ)}{dx} &= -2(w\eta - v\zeta),\\
\frac{d(P-kQ)}{dy} &= -2(u\zeta - w\xi),\\
\frac{d(P-kQ)}{dz} &= -2(v\xi - u\eta).
\end{aligned}\right\} \tag{13}$$

Multiplying these by u, v, w respectively, and then by ξ, η, ζ, and in each case adding the results, we have, writing for brevity

$$\Theta = P - kQ,$$

$$\left.\begin{aligned}
u\frac{d\Theta}{dx} + v\frac{d\Theta}{dy} + w\frac{d\Theta}{dz} &= 0,\\
\xi\frac{d\Theta}{dx} + \eta\frac{d\Theta}{dy} + \zeta\frac{d\Theta}{dz} &= 0,
\end{aligned}\right\} \tag{14}$$

and also

$$\frac{d\Theta}{dn} = q'\Omega \sin \delta, \tag{15}$$

where $q' = \sqrt{2}\,q$ is the current velocity, and δ is the angle between the stream line and the vortex line at the point x, y, z. Hence the conditions that the state of motion of the fluid for which

$$\Delta^2 u \,.\, dx + \Delta^2 v \,.\, dy + \Delta^2 w \,.\, dz$$

is an exact differential, are as follows: It must be possible to draw in the fluid a system of surfaces, $\Theta =$ const., infinite in number, and each of which is covered by a network of stream lines and vortex lines. This is the property denoted by equations (14). The product $q'\Omega \sin \delta \, dn$ must be constant over each such surface, dn denoting the length of the normal drawn to the consecutive surface of the system. These results are identical in form with those given for a perfect fluid by Professor Lamb in his work on Fluid Motion. In order that

$$\Delta^2 u \,.\, dx + \Delta^2 v \,.\, dy + \Delta^2 w \,.\, dz$$

shall be an exact differential, the equations of condition

$$\Delta^2 \xi = 0, \qquad \Delta^2 \eta = 0, \qquad \Delta^2 \zeta = 0, \tag{16}$$

must hold.

If we assume that the motion of the fluid is so slow that squares and products of the velocities may be neglected, equations (1) become, when a potential exists,

$$\begin{aligned} k\Delta^2 u &= \frac{dU}{dx}, \\ k\Delta^2 v &= \frac{dU}{dy}, \\ k\Delta^2 w &= \frac{dU}{dz}, \end{aligned} \tag{17}$$

where

$$U = V + \int \frac{dp}{\rho}. \tag{18}$$

In this case the quantity

$$\Delta^2 u \,.\, dx + \Delta^2 v \,.\, dy + \Delta^2 w \,.\, dz$$

is obviously an exact differential, and the equations of condition

$$\Delta^2 \xi = 0, \qquad \Delta^2 \eta = 0, \qquad \Delta^2 \zeta = 0,$$

are satisfied.

From equations (12) we have, in every case,

$$
\begin{aligned}
u\left(\frac{dP}{dx}-k\Delta^2u\right)+v\left(\frac{dP}{dy}-k\Delta^2v\right)+w\left(\frac{dP}{dz}-k\Delta^2w\right)&=0,\\
\xi\left(\frac{dP}{dx}-k\Delta^2u\right)+\eta\left(\frac{dP}{dy}-k\Delta^2v\right)+\zeta\left(\frac{dP}{dz}-k\Delta^2w\right)&=0,
\end{aligned}
\tag{19}
$$

so that the conditions for steady motion hold now as in the particular case just mentioned; but in this case the surfaces in the fluid are given by the differential equation

$$dP-k\left(\Delta^2u\,.\,dx+\Delta^2v\,.\,dy+\Delta^2w\,.\,dz\right)=0. \tag{20}$$

Write for convenience

$$
\begin{aligned}
L&=v\zeta-w\eta,\\
M&=w\xi-u\zeta,\\
N&=u\eta-v\xi.
\end{aligned}
\tag{21}
$$

Equations (12) now become

$$
\begin{aligned}
\frac{dP}{dx}-2\,L&=k\Delta^2u,\\
\frac{dP}{dy}-2\,M&=k\Delta^2v,\\
\frac{dP}{dz}-2\,N&=k\Delta^2w;
\end{aligned}
\tag{22}
$$

and from these, by differentiating for x, y, z, respectively, and adding, we have

$$\Delta^2P=2\left(\frac{dL}{dx}+\frac{dM}{dy}+\frac{dN}{dz}\right). \tag{23}$$

The same equation holds when the motion is not steady; for if we differentiate equations (10) for x, y, z, respectively, and add, the terms containing the differential coefficients of u, v, w with respect to t will disappear by virtue of the equation of continuity. Integrating (23), and substituting for P its value, there results

$$\frac{p}{\rho}=G-\left\{\frac{1}{2\pi}\iiint\left(\frac{dL}{dx'}+\frac{dM}{dy'}+\frac{dN}{dz'}\right)\frac{dx'dy'dz'}{r}+V+q\right\}. \tag{24}$$

In the general case G is a function of the time; for steady motion, however, it is a constant. The quantities L, M, N may be the first differential coefficients with respect to x, y, z of function of x, y, z; suppose such a function Ψ to exist that we have

$$
\begin{aligned}
2\,L&=\frac{d\Psi}{dx},\\
2\,M&=\frac{d\Psi}{dy},\\
2\,N&=\frac{d\Psi}{dz},
\end{aligned}
\tag{25}
$$

equations (22) become in this case

$$\frac{d(P-\Psi)}{dx} = k\Delta^2 u,$$
$$\frac{d(P-\Psi)}{dy} = k\Delta^2 v, \qquad (26)$$
$$\frac{d(P-\Psi)}{dz} = k\Delta^2 w;$$

and from these results

$$\Delta^2(P-\Psi) = 0, \qquad (27)$$

and consequently

$$\frac{p}{\rho} = G - \left\{\frac{1}{4\pi}\iiint \frac{\Delta_1^2\Psi}{r}\,dx'dy'dz' + V + q\right\}, \qquad (28)$$

in which

$$\Delta_1^2 \equiv \frac{d^2}{dx'^2} + \frac{d^2}{dy'^2} + \frac{d^2}{dz'^2}. \qquad (29)$$

Equation (24) can be thrown into another form by very simple transformations. Write

$$D^2 = L^2 + M^2 + N^2; \qquad (30)$$

then, denoting by α, β, γ the direction-cosines of the vector D,

$$L = \alpha D,$$
$$M = \beta D, \qquad (31)$$
$$N = \gamma D.$$

Substituting in (30) the values of L, M, N, we have

$$D = q'\Omega\left[1 - \left(\frac{u\xi}{q'\Omega} + \frac{v\eta}{q'\Omega} + \frac{w\zeta}{q'\Omega}\right)^2\right], \qquad (32)$$

or

$$D = q'\Omega \sin\delta, \qquad (33)$$

where δ is the angle between the stream line and the vortex line at the point x, y, z. Now let a, b, c denote the direction-cosines of a normal to the closed surface containing the fluid; then

$$a\alpha + b\beta + c\gamma$$

is the sine of the angle between the normal and the plane containing the instantaneous axis of rotation Ω and the direction of the resultant velocity q', or

$$\sin\phi = a\alpha + b\beta + c\gamma. \qquad (34)$$

Similarly, if we denote by a', b', c' the direction-cosines of the line joining (x, y, z)

to (x', y', z') and ϕ' the angle between this line and the above-mentioned plane, we have

$$\sin\phi' = a'\alpha + b'\beta + c'\gamma. \tag{35}$$

Take now the triple integral in (24),

$$-\frac{1}{2\pi}\iiint\left(\frac{dL}{dx'}+\frac{dM}{dy'}+\frac{dN}{dz'}\right)\frac{dx'dy'dz'}{r};$$

this is

$$=\frac{1}{2\pi}\iint(aL+bM+cN)\frac{d\sigma}{r}+\frac{1}{2\pi}\iiint\frac{a'L+b'M+c'N}{r^2}dx'dy'dz',$$

where $d\sigma$ is an element of the bounding surface. By virtue of the above equations, we have

$$aL+bM+cN=(a\alpha+b\beta+c\gamma)\,D=q'\Omega\sin\delta\sin\phi,$$

and similarly

$$a'L+b'M+c'N=q'\Omega\sin\delta\sin\phi';$$

therefore

$$\left.\begin{aligned}&-\frac{1}{2\pi}\iiint\left(\frac{dL}{dx'}+\frac{dM}{dy'}+\frac{dN}{dz'}\right)\frac{dx'dy'dz'}{r}\\&=\frac{1}{2\pi}\iint\frac{q'\Omega\sin\delta\sin\phi}{r}d\sigma+\frac{1}{2\pi}\iiint\frac{q'\Omega\sin\delta\sin\phi'}{r^2}dx'dy'dz',\end{aligned}\right\} \tag{36}$$

and finally (24) becomes

$$\frac{p}{\rho}=G-(V+q)+\frac{1}{2\pi}\iint\frac{q'\Omega\sin\delta\sin\phi}{r}d\sigma+\frac{1}{2\pi}\iiint\frac{q'\Omega\sin\delta\sin\phi'}{r^2}dx'dy'dz'. \tag{37}$$

If the motion in the fluid is a screw motion, i. e. if the direction of motion be along the instantaneous axis of rotation, we shall have $\delta=0$, and consequently

$$\frac{p}{\rho}=G-(V+q). \tag{38}$$

If the plane containing the direction of motion and the instantaneous axis of rotation be always normal to the bounding surface, we shall have $\phi=0$, and then

$$\frac{p}{\rho}=G-(V+q)+\frac{1}{2\pi}\iiint\frac{q'\Omega\sin\delta\sin\phi'}{r^2}dx'dy'dz'. \tag{39}$$

We will pass now to the consideration of one or two particular cases. Assume, first, that u, v, w are given by the equations

$$u = \frac{dW}{dy} - \frac{dV}{dz},$$
$$v = \frac{dU}{dz} - \frac{dW}{dx}, \qquad (40)$$
$$w = \frac{dV}{dx} - \frac{dU}{dy}.$$

The functions U, V, W must, as is well known, satisfy the equations of condition

$$\Delta^2 U = -2\xi, \qquad \Delta^2 V = -2\eta, \qquad \Delta^2 W = -2\zeta,$$
$$\frac{dU}{dx} + \frac{dV}{dy} + \frac{dW}{dz} = 0. \qquad (41)$$

Instead of the three functions U, V, W, we may introduce a single function Φ and write

$$U = z\frac{d\Phi}{dy} - y\frac{d\Phi}{dz},$$
$$V = x\frac{d\Phi}{dz} - z\frac{d\Phi}{dx}, \qquad (42)$$
$$W = y\frac{d\Phi}{dx} - x\frac{d\Phi}{dy};$$

these quantities will satisfy equations (41), and give us for the values of ξ, η, ζ,

$$\xi = \frac{1}{2}\left(z\frac{d\phi}{dy} - y\frac{d\phi}{dz}\right),$$
$$\eta = \frac{1}{2}\left(x\frac{d\phi}{dz} - z\frac{d\phi}{dx}\right), \qquad (43)$$
$$\zeta = \frac{1}{2}\left(y\frac{d\phi}{dx} - x\frac{d\phi}{dy}\right),$$

where

$$\phi = -\Delta^2\Phi.$$

The function ϕ must also satisfy the equation

$$\Delta^2\phi = 0, \qquad (45)$$

since

$$\Delta^2\xi = \Delta^2\eta = \Delta^2\zeta = 0.$$

For ϕ we can take any homogeneous function of the n^{th} degree satisfying (45) and then will have

$$x\frac{d\phi}{dx} + y\frac{d\phi}{dy} + z\frac{d\phi}{dz} = n\phi, \qquad (46)$$

or ϕ represents a solid spherical harmonic of the degree n. For the values of u, v, w we have now

$$
\begin{aligned}
u &= \frac{d}{dx}\left\{\Phi + x\frac{d\Phi}{dx} + y\frac{d\Phi}{dy} + z\frac{d\Phi}{dz}\right\} + x\phi, \\
v &= \frac{d}{dy}\left\{\Phi + x\frac{d\Phi}{dx} + y\frac{d\Phi}{dy} + z\frac{d\Phi}{dz}\right\} + y\phi, \\
w &= \frac{d}{dz}\left\{\Phi + x\frac{d\Phi}{dx} + y\frac{d\Phi}{dy} + z\frac{d\Phi}{dz}\right\} + z\phi.
\end{aligned} \tag{47}
$$

Reverting to equations (3) for a convenient form of obtaining the values of $\Delta^2 u$, $\Delta^2 v$, $\Delta^2 w$, we readily find for these quantities the values

$$
\begin{aligned}
\Delta^2 u &= n\frac{d\phi}{dx}, \\
\Delta^2 v &= n\frac{d\phi}{dy}, \\
\Delta^2 w &= n\frac{d\phi}{dz}.
\end{aligned} \tag{48}
$$

The function Q of equations (13) is now the function $n\phi$, or Q is in this case a solid spherical harmonic of the degree n. Equations (40) involve the assumption that the motion is purely of a rotational character. If we for a moment abstract the friction in the fluid from consideration, the motion, if caused by a solid moving in the fluid, will be irrotational, and therefore subject to a velocity potential, say ψ, satisfying the equation $\Delta^2\psi = 0$. We can then in general write the values of u, v, w in the forms

$$
\begin{aligned}
u &= -\frac{d\psi}{dx} + \frac{dW}{dy} - \frac{dV}{dz}, \\
v &= -\frac{d\psi}{dy} + \frac{dU}{dz} - \frac{dW}{dx}, \\
w &= -\frac{d\psi}{dz} + \frac{dV}{dx} - \frac{dU}{dy}.
\end{aligned} \tag{49}
$$

Equations (43), giving the values of the rotation components, will be, of course, unaltered by this change in the values of u, v, w, and we shall have, instead of (47),

$$
\begin{aligned}
u &= \frac{d}{dx}\left\{-\psi + \Phi + x\frac{d\Phi}{dx} + y\frac{d\Phi}{dy} + z\frac{d\Phi}{dz}\right\} + x\phi, \\
v &= \frac{d}{dy}\left\{-\psi + \Phi + x\frac{d\Phi}{dx} + y\frac{d\Phi}{dy} + z\frac{d\Phi}{dz}\right\} + y\phi, \\
w &= \frac{d}{dz}\left\{-\psi + \Phi + x\frac{d\Phi}{dx} + y\frac{d\Phi}{dy} + z\frac{d\Phi}{dz}\right\} + z\phi.
\end{aligned} \tag{50}
$$

From equations (43) we derive at once

$$\xi \frac{d\phi}{dx} + \eta \frac{d\phi}{dy} + \zeta \frac{d\phi}{dz} = 0, \tag{51}$$

from which it follows that the vortex lines lie on the surfaces given by the equation $\phi = \text{const.}$ From equations (14) we have, however, since in this case

$$\Delta^2 u \, . \, dx + \Delta^2 v \, . \, dy + \Delta^2 w \, . \, dz$$

is an exact differential,

$$\xi \frac{d\Theta}{dx} + \eta \frac{d\Theta}{dy} + \zeta \frac{d\Theta}{dz} = 0,$$

and consequently the vortex lines in the fluid lie at the intersection of the surfaces

$$\left. \begin{aligned} \phi &= \text{const.} \\ \Theta &= \text{const.} \end{aligned} \right\} \tag{52}$$

The surfaces Θ are fixed in the fluid, but the surfaces ϕ may move; their motion, however, will always be in such a manner that the above condition shall be satisfied. Equations (50) can be thrown into a simpler form by the following considerations. Write

$$r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$$

and

$$\log r = \lambda.$$

Then

$$x \frac{d\Phi}{dx} + y \frac{d\Phi}{dy} + z \frac{d\Phi}{dz} = r \frac{d\Phi}{dr};$$

but

$$\frac{d}{dr} = \frac{1}{r} \frac{d}{d\lambda},$$

and consequently

$$r \frac{d\Phi}{dr} = \frac{d\Phi}{d\lambda}.$$

We have then

$$\left. \begin{aligned} u &= \frac{x}{r^2} \frac{d}{d\lambda} \left(\Phi - \psi + \frac{d\Phi}{d\lambda} \right) + x\phi, \\ v &= \frac{y}{r^2} \frac{d}{d\lambda} \left(- \psi + \Phi + \frac{d\Phi}{d\lambda} \right) + y\phi, \\ w &= \frac{z}{r^2} \frac{d}{d\lambda} \left(- \psi + \Phi + \frac{d\Phi}{d\lambda} \right) + z\phi. \end{aligned} \right\} \tag{53}$$

The solution of the problem when the motion of the fluid is caused by a sphere moving through it is quite simple.* We have first to determine the velocity

* See an article on this subject by the author in the "Philosophical Magazine" for November, 1880.

potential. In Lamb's "Treatise on Fluid Motion" he gives a solution, due to Stokes, of the problem of a sphere moving with uniform velocity in a viscous fluid in the case when the motion of the solid is along the axis of x, and the motion of the fluid is symmetrical around this axis. Special polar co-ordinates are employed in obtaining the required solution, but from the general values above given for u, v, w, we can readily obtain the same results in a very simple manner. We will consider this case for a moment, as the forms of u, v, w thus obtained are of use in another and rather more difficult problem. At an infinitely great distance from the origin the fluid is streaming along the axis of x with a velocity $= -\lambda$, or

$$u = -\lambda,$$

$$v = 0,$$

$$w = 0,$$

for all of the motion which is due to a velocity potential. We have then, as indeed we know from other considerations,

$$\phi = -\lambda x.$$

Our former value of ϕ was

$$\phi = \sum_0^\infty L_i \phi_i,$$

where

$$L_i = 1 + \frac{i}{i+1}\left(\frac{a}{r}\right)^{2i+1};$$

this will now reduce to

$$\phi = \phi_1 = -\lambda x.$$

For the general value of u we take into account the friction terms. The quantities L_i and R_i will all disappear with the exception of L_1 and R_1, and these also vanish at infinity. For these we have the values

$$L_1 = -1 + \frac{1}{2}\left(\frac{a}{r}\right)^3,$$

$$R_1 = -\frac{3}{4}\left(\frac{a}{r}\right)^3 + \frac{3}{4}\frac{a}{r}.$$

For u we have, then,

$$u = \frac{d\phi_1}{dx}\left[L_1 + 2R_1 + r\frac{dR_1}{dr}\right] + \phi_1 \frac{d}{dx}(L_1 - R_1),$$

which, on substitution of the values of L_1, R_1, ϕ_1, becomes

$$u = \lambda\left[1 - \frac{3}{4}\frac{a}{r} - \frac{1}{4}\left(\frac{a}{r}\right)^3\right] - \frac{3\lambda}{4}\left[\frac{a}{r^3} - \frac{a^3}{r^5}\right]x^2;$$

or

$$u = \lambda\left[1 - \frac{3}{4}\frac{a}{r}\right] - \frac{3\lambda a}{4}\frac{x^2}{r^3} - \frac{\lambda a^3}{4}\cdot\left[\frac{r^2 - 3x^2}{r^5}\right].$$

But

$$\frac{r^2 - 3x^2}{r^5} = \frac{d^2}{dx^2}\cdot\frac{1}{r},$$

and

$$\frac{x^2}{r^3} = x\frac{d}{dx}\frac{1}{r};$$

Writing then

$$\frac{3\lambda a}{4}\cdot\frac{1}{r} = \chi_1,$$

$$\frac{\lambda a^3}{4}\cdot\frac{1}{r} = \chi_2,$$

we have finally

$$u = \lambda - \chi_1 + x\frac{d\chi_1}{dx} + \frac{d^2\chi_2}{dx^2},$$

or

$$u = (\lambda - 2\chi_1) + \frac{d}{dx}\left(x\chi_1 + \frac{d\chi_2}{dx}\right).$$

Similarly,

$$v = \frac{d}{dy}\left(x\chi_1 + \frac{d\chi_2}{dx}\right),$$

$$w = \frac{d}{dz}\left(x\chi_1 + \frac{d\chi_2}{dx}\right).$$

The determination of the resistance experienced by the sphere, supposing it to move along x with velocity $= +\lambda$, is the same thing as the determination of the pressure upon the sphere supposed at rest and the fluid streaming past it with velocity $= -\lambda$. The latter case is the one that we are dealing with, and to obtain this pressure we use the dissipation-function, employing the form

$$E = 2\mu\{\Delta^2 q - (u\Delta^2 u + v\Delta^2 v + w\Delta^2 w) - 2\Omega^2\}.$$

We had

$$2q = u^2 + v^2 + w^2,$$

then

$$2\Delta^2 q = \Delta^2(u^2 + v^2 + w^2);$$

substituting the values of u, v, w, gives

$$2q = (\lambda - 2\chi_1)^2 + 2(\lambda - 2\chi_1)\frac{dQ}{dx} + \left(\frac{dQ}{dx}\right)^2 + \left(\frac{dQ}{dy}\right)^2 + \left(\frac{dQ}{dz}\right)^2,$$

where for brevity we have written

$$Q = x\chi + \frac{d\chi_2}{dx};$$

to this add

$$a = \frac{3\lambda a}{4},$$

$$\rho = \frac{\lambda a^3}{4};$$

now, introducing the values of Q and X_1, we find readily

$$2\,\Delta^2 q = a^2\,\frac{8\,r^2 - 12\,x^2}{r^8} + \beta^2\,\frac{36\,r^2 + 72\,x^2}{r^{10}} + a\beta\,\frac{4\,r^2 - 60\,x^2}{r^8} + 4\,a\lambda\,\frac{3\,x^2 - r^2}{r^5}.$$

We also find easily

$$2\,(u\Delta^2 u + v\Delta^2 v + w\Delta^2 w) = a^2\,\frac{4\,r^2 - 20\,x^2}{r^8} + a\beta\,\frac{12\,x^2 + 4\,r^2}{r^8} + 4a\lambda\,\frac{3\,x^2 - r^2}{r^5}$$

and

$$4\,\Omega^2 = 4\,a^2\,\frac{r^2 - x^2}{r^8}.$$

Combining all of these, we obtain

$$\begin{aligned} E &= \quad \mu\left\{a^2\,\frac{12\,x^2}{r^8} + \beta^2\,\frac{36\,r^2 + 72\,x^2}{r^{10}} - 2\,a\beta\,\frac{36\,x^2}{r^8}\right\} \\ &= 12\,\mu\left\{\frac{a^2}{r^4} + \frac{9\,\beta^2}{r^8} - \frac{6\,a\beta}{r^6}\right\}\frac{x^2}{r^2} - \mu\,\frac{36\,\beta^2 x^2}{r^{10}} + \mu\,\frac{36\,\beta^2 r^2}{r^{10}} \\ &= 12\,\mu\left\{\frac{a}{r^2} - \frac{3\,\beta}{r^4}\right\}^2\frac{x^2}{r^2} + \mu\,\frac{36\,\beta^2}{r^8}\left\{1 - \frac{x^2}{r^2}\right\}. \end{aligned}$$

Writing

$$\frac{x}{r} = \cos\theta,$$

multiplying E by $2\,\pi r^2 \sin\theta\, d\theta dr$, and integrating from $\theta = 0$ to $\theta = \pi$ and from $r = a$ to $r = \infty$, we have, for the total rate of dissipation of energy,

$$16\,\pi\mu\left\{\frac{3\,\beta^2}{a^5} - \frac{2\,a\beta}{a^3} + \frac{a^2}{a}\right\} = 6\,\pi\mu a\lambda^2.$$

If X denote the force which must act upon the sphere in order to keep it at rest, we have

$$X\lambda = 6\,\pi\mu a\lambda^2,$$

or

$$X = 6\,\pi\mu a\lambda.$$

These are the results given by Lamb in his treatise. In the general case the velocity u can be written in the form

$$u = \sum_0^\infty\left[\frac{d\phi_i}{dx}\left\{(2\,i + 1)\,R_i + r\,\frac{dR_i}{dr}\right\} - \frac{d}{dx}\,(L_i\phi_i) - i\,\frac{d}{dx}\,(R_i\phi_i)\right]$$

with similar expressions for v and w. Applying the operator Δ^2 to this, we have, if we assume

$$u = u_1 + u_2 + \ldots + u_i,$$

$$\Delta^2 u_i = i\,\frac{ds_i}{dx},$$

and

$$\Delta^2 u = \frac{d}{dx}\,\Sigma s_i;$$

also

$$\Delta^2 v = \frac{d}{dy}\,\Sigma s_i,$$

$$\Delta^2 w = \frac{d}{dz}\,\Sigma s_i.$$

The other terms vanish on applying this operator, as of course they should do.

In the case where the axis of spin and the direction of the current velocity lie always in a plane normal to the surface of the solid, we have for the determination of the pressure

$$\frac{p}{\rho} = G - (V + q) + \frac{1}{2\pi}\iiint \frac{q'\Omega \sin\delta \sin\phi'}{r^2}\,dx'dy'dz'.$$

In the problem just discussed of the motion of the fluid all parallel to the axis of x, we have

$$u\xi + v\eta + w\zeta = 0,$$

or the stream lines and vortex lines are at right angles to each other; this gives

$$\sin\delta = 1,$$

and obviously in this case the plane above mentioned is normal to the surface of the sphere. We have then

$$\frac{p}{\rho} = G - (V + q) + \frac{1}{2\pi}\iiint \frac{q'\Omega \sin\phi'}{r^2}\,dx'dy'dz'.$$

Concerning V we have

$$-\frac{dV}{dx} = X,$$

therefore

$$V = 6\,\pi\mu a\lambda x + \text{const.}$$

In the general case, where there is no restriction as to the direction of motion of the fluid, the pressure must be determined by means of the equation

$$\frac{p}{\rho} = G - (V + q) + \frac{1}{2\pi}\iint \frac{q'\Omega \sin\delta \sin\phi}{r}\, d\sigma + \frac{1}{2\pi}\iiint \frac{q'\Omega \sin\delta \sin\phi'}{r^2}\, dx'dy'dz'.$$

The computation would certainly be very difficult, if not impossible, from the complicated nature of the quantities involved. The first step, however, would be the determination of V; the angles δ and ϕ can be found from the expressions for the velocities. If the motion is very slow and no external forces act on the fluid, we shall always have

$$\frac{p}{\rho} = \int (\Delta^2 u \,.\, dx + \Delta^2 v \,.\, dy + \Delta^2 w \,.\, dz).$$

Kirchhoff (*vide* Mathematische Physik, p. 377) has solved the problem of an ellipsoid of revolution rotating with constant velocity about its axis in a viscous fluid, both for the cases of an infinite extent of fluid and for a mass of fluid contained within a confocal ellipsoid. I do not see how to attack the general problem of the motion of any ellipsoid in a mass of viscous fluid; but for the case of simple translation along one of the axes it is not difficult to find values for u, v, w which will satisfy all the prescribed conditions. Suppose an ellipsoid in the fluid with its axes coinciding with those of the co-ordinates,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

If we assume first the case of no friction, and call V the potential of the ellipsoid at an external point, we have

$$V = \pi abc \int_\sigma^\infty \frac{1 - \dfrac{x^2}{a^2+\psi} - \dfrac{y^2}{b^2+\psi} - \dfrac{z^2}{c^2+\psi}}{\sqrt{(a^2+\psi)(b^2+\psi)(c^2+\psi)}}\, d\psi,$$

or

$$V = \text{const.} - 2\pi(Ax^2 + By^2 + Cz^2),$$

where σ is the greatest root of the equation

$$\frac{x^2}{a^2+\sigma} + \frac{y^2}{b^2+\sigma} + \frac{z^2}{c^2+\sigma} = 1.$$

Now for the velocity potential ϕ we have (American Journal of Mathematics, Vol. II. p. 260, et seq.),

$$\phi = -\lambda\left(x - \frac{1}{2\pi(2-A)}\frac{dV}{dx}\right).$$

The quantities A, B, C are known to be

$$A = abc \int_0^\infty \frac{d\psi}{(a^2 + \psi) N},$$

$$B = abc \int_0^\infty \frac{d\psi}{(b^2 + \psi) N},$$

$$C = abc \int_0^\infty \frac{d\psi}{(c^2 + \psi) N},$$

in which

$$N = \sqrt{(a^2 + \psi)(b^2 + \psi)(c^2 + \psi)}.$$

The velocities u, v, w, in this case, will have for values

$$u = -\lambda + \frac{1}{2\pi(2 - A)} \frac{d^2 V}{dx^2},$$

$$v = \frac{1}{2\pi(2 - A)} \frac{d^2 V}{dxdy},$$

$$w = \frac{1}{2\pi(2 - A)} \frac{d^2 V}{dxdz}.$$

Reverting now for a moment to the case of the sphere, we had

$$u = \lambda - 2\chi_1 + \frac{d}{dx}(x\chi_1) + \frac{d^2\chi_2}{dx^2},$$

$$v = \frac{d}{dy}(x\chi_1) + \frac{d^2\chi_2}{dxdy},$$

$$w = \frac{d}{dz}(x\chi_1) + \frac{d^2\chi_2}{dxdz},$$

in which

$$\chi_2 = \frac{\lambda a^3}{4} \cdot \frac{1}{r},$$

a quantity proportional to the potential of the solid homogeneous sphere upon an external point. The same remark, of course, holds concerning $\frac{V}{2\pi(2 - A)}$. Write now for the velocities in the case of the ellipsoid the following system of values similar to those obtained for the sphere:—

$$u = \lambda - 2\Psi + \frac{d}{dx}(x\Psi) + \frac{1}{2\pi(2 - A)} \frac{d^2 V}{dx^2},$$

$$v = \frac{d}{dy}(x\Psi) + \frac{1}{2\pi(2 - A)} \frac{d^2 V}{dxdy},$$

$$w = \frac{d}{dz}(x\Psi) + \frac{1}{2\pi(2 - A)} \frac{d^2 V}{dxdz}.$$

These must satisfy the equation

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0;$$

this gives

$$-2\frac{d\Psi}{dx} + \Delta^2(x\Psi) = 0,$$

or simply

$$\Delta^2\Psi = 0.$$

This is satisfied (Ferrer's Spherical Harmonics, p. 110) by assuming

$$\Psi = \int_\sigma^\infty \frac{d\psi}{\sqrt{(a^2+\psi)(b^2+\psi)(c^2+\psi)}},$$

ψ and σ having the meaning already assigned them. For greater convenience we will write

$$V = \text{const.} - (A_1x^2 + B_1y^2 + C_1z^2),$$

where

$$A_1 = 2\pi A, \text{ etc.},$$

also

$$\Psi = 2\pi abc\int_\sigma^\infty \frac{d\psi}{\sqrt{(a^2+\psi)(b^2+\psi)(c^2+\psi)}}.$$

This (Ferrers, p. 111) is the potential of a homogeneous ellipsoidal shell of determinate density at an external point. For the density we have (Kirchhoff, p. 179),

$$\frac{d\Psi}{dn_i} + \frac{d\Psi}{dn_\sigma} = -4\pi h,$$

which gives at once

$$h = \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{-1};$$

that is, the density of the ellipsoidal shell at any point is proportional to the central perpendicular upon the tangent plane to the surface at that point. The values given now for u, v, w satisfy all required conditions, and we have in this case also

$$\Delta^2u\,.\,dx + \Delta^2v\,.\,dy + \Delta^2w\,.\,dz,$$

an exact differential, viz. $d\,.\,\Delta^2(x\Psi)$. For greater generality, however, we may introduce two arbitrary constants, say α and β; then

$$u = \lambda - 2\alpha\Psi + \alpha\frac{d}{dx}(x\Psi) + \frac{\beta}{(4\pi - A_1)}\frac{d^2V}{dx^2},$$

$$v = \alpha\frac{d}{dy}(x\Psi) + \frac{\beta}{(4\pi - A_1)}\frac{d^2V}{dxdy},$$

$$w = \alpha\frac{d}{dz}(x\Psi) + \frac{\beta}{(4\pi - A_1)}\frac{d^2V}{dxdz}.$$

Now at the surface of the body we have

$$u = v = w = 0,$$

and also at the surface $\sigma = 0$; therefore

$$\Psi_0 = 2\pi abc \int_0^\infty \frac{d\psi}{\sqrt{(a^2+\psi)(b^2+\psi)(c^2+\psi)}}.$$

It will be convenient here to make a little digression and give the values of certain of our quantities as elliptic functions. Take ψ_1, ψ_2, ψ_3 as the variable parameters of a system of surfaces confocal to the given ellipsoid; $\delta_1, \delta_2, \delta_3$ as the amplitudes of three elliptic integrals

$$\theta_1 = \int \frac{d\delta_1}{\Delta(k_1\delta_1)}, \quad \theta_2 = \int \frac{d\delta_2}{\Delta(k_1\delta_2)}, \quad \theta_3 = \int \frac{d\delta_3}{\Delta(k_1\delta_3)}.$$

We have now

$$x^2 = \frac{(a^2+\psi_1)(a^2+\psi_2)(a^2+\psi_3)}{(a^2-b^2)(a^2-c^2)},$$

$$y^2 = \frac{(b^2+\psi_1)(b^2+\psi_2)(b^2+\psi_3)}{(b^2-c^2)(b^2-a^2)},$$

$$z^2 = \frac{(c^2+\psi_1)(c^2+\psi_2)(c^2+\psi_3)}{(c^2-a^2)(c^2-b^2)}.$$

Write now

$$\psi_1 = c^2 \frac{\frac{a^2-c^2}{c^2} - t^2}{x^2},$$

then make

$$t = \tan \delta_1,$$

where δ_1 lies between 0 and $\frac{\pi}{2}$. Similar transformations for ψ_2 and ψ_3 give us finally

$$x = \sqrt{a^2-c^2} \cdot \frac{dn\,\theta_2\, sn\,\theta_3}{sn\,\theta_1},$$

$$y = \sqrt{a^2-c^2}\, \frac{dn\,\theta_1\, cn\,\theta_2\, cn\,\theta_3}{sn\,\theta_1},$$

$$z = \sqrt{a^2-c^2}\, \frac{cn\,\theta_1\, sn\,\theta_2\, dn\,\theta_3}{sn\,\theta_1}.$$

In the above the modulus k is $= \sqrt{\frac{a^2-b^2}{a^2-c^2}}$.

We have also for the quantities A_1, B_1, C_1 the values

$$A_1 = \frac{2\,abc}{(a^2 - c^2)^{\frac{1}{2}}} \int_0^{\frac{\pi}{2}} \operatorname{sn}^2 \theta_1 \, d\theta_1,$$

$$B_1 = \frac{2\,abc}{(a^2 - c^2)^{\frac{1}{2}}} \int_0^{\frac{\pi}{2}} \left(\frac{1 - \operatorname{dn}^2 \theta_1}{\operatorname{dn}^2 \theta_1}\right) d\theta_1,$$

$$C_1 = \frac{2\,abc}{(a^2 - c^2)^{\frac{1}{2}}} \int_0^{\frac{\pi}{2}} \frac{\operatorname{sn}^2 \theta_1}{\operatorname{cn}^2 \theta_1} \, d\theta_1;$$

or

$$A_1 = -\frac{I}{k^2}\left\{\frac{k\theta_1}{K}\frac{dE}{dk} + \frac{\Theta'(\theta_1)}{\Theta(\theta_1)}\right\},$$

$$B_2 = \frac{I}{k^2}\left\{\frac{\theta_1}{k}\frac{d \log K}{dk} + \frac{1}{k'^2}\frac{\Theta'(\theta_1 + K)}{\Theta(\theta_1 + K)}\right\},$$

$$C_2 = \frac{I}{k'^2}\left\{\frac{d}{d\theta_1} \log H(\theta_1 + K) - \frac{E}{K}\theta_1\right\},$$

where for brevity I have written

$$I = \frac{2\,abc}{(a^2 - c^2)^{\frac{1}{2}}}.$$

The above transformations are given in full in an article, "On the Motion of an Ellipsoid in a Fluid," American Journal of Mathematics, Vol. II. We find, by the same transformations,

$$\Psi = 2\,\pi abc \int_\sigma^\infty \frac{d\psi_1}{N_1} = \frac{4\,\pi abc}{\sqrt{a^2 - c^2}}\,\theta_1,$$

and also

$$\Psi_0 = 2\,\pi abc \int_0^\infty \frac{d\psi_1}{N_1} = \frac{4\,\pi abc}{\sqrt{a^2 - c^2}}\,K,$$

K denoting the complete elliptic integral of the first kind. Equating now to zero the found values of u, v, w, we have at once, by means of the foregoing transformations,

$$\alpha = \frac{(a^2 - c^2)^{\frac{1}{2}}}{2\,abc} \cdot \frac{\lambda}{a^2 \int_0^{\frac{\pi}{2}} \operatorname{sn}^2 \theta_1 \, d\theta_1 - 2\pi (a^2 - c^2) K}$$

and

$$\beta = a^2 \alpha.$$

The computation of the force necessary to keep the body in place, or the force which must be applied to overcome the resistance of the fluid if the body is moving through it, could be in this case, as in the former, computed by means of the dissipation-function; but the process would be rather tedious and complicated. We can, however, without much difficulty, compute the pressure over the surface of the body supposed in motion. Observe that since Ψ is a surface potential corresponding to a surface density

$$\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{-\frac{1}{2}}=\omega,$$

at each point of the ellipsoidal shell we can write

$$\Psi=\iint\frac{\omega}{r'}\,d\epsilon',$$

where $d\epsilon'$ is an element of area of the surface, and

$$r'=\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}.$$

If we denote by p_1, p_2, p_3 the pressures on unit of area, and by $d\epsilon$ the element of area of an infinitely great sphere containing the fluid, the following equations are known to be satisfied:

$$\iint p_1 d\epsilon+\iint p_1 d\epsilon'=0,$$
$$\iint p_2 d\epsilon+\iint p_2 d\epsilon'=0,$$
$$\iint p_3 d\epsilon+\iint p_3 d\epsilon'=0;$$

in our case these reduce to the first equation only since $p_2=p_3=0$.

Making $r_1^2=x'^2+y'^2+z'^2$, we have

$$\frac{1}{r'}=\frac{1}{\sqrt{r^2+r_1^2-2rr_1}}=\frac{1}{r}\cdot\frac{1}{\sqrt{1+\frac{r_1^2}{r^2}-\frac{2r_1}{r}}}$$

for the points at infinity, i. e. the points on the sphere it will be sufficient to take

$$\frac{1}{r'}=\frac{1}{r},$$

and consequently

$$\Psi=\frac{1}{r}\iint\omega d\epsilon'.$$

For brevity, write

$$N_2=\sqrt{(a^2+\psi_2)(b^2+\psi_2)(c^2+\psi_2)},$$
$$N_3=\sqrt{(a_2+\psi_3)(b^2+\psi_3)(c^2+\psi_3)}.$$

The quantity $\omega d\epsilon'$ is now given by (Ferrer's Spherical Harmonics, page 129)

$$\omega d\epsilon'=\frac{iabc}{4}\frac{(\psi_3-\psi_2)}{N_2N_3}\,d\psi_2 d\psi_3,$$

and consequently

$$\Psi=\frac{iabc}{4}\iint\frac{(\psi_3-\psi_2)}{N_2N_3}\,d\psi_2 d\psi_3,$$

or, since ψ_2 is between $-c_2$ and $-b_2$ and ψ_3 between $-b_2$ and $-a_2$,

$$\iint\omega d\epsilon'=\frac{iabc}{4}\left\{\int_{a^2}^{-b^2}\frac{\psi_3 d\psi_3}{N_3}\int_{-b^2}^{-c^2}\frac{d\psi_2}{N_2}-\int_{-a^2}^{-b^2}\frac{d\psi_3}{N_3}\int_{-b^2}^{c^2}\frac{\psi_2 d\psi_2}{N}\right\}.$$

For N_2 and N_3 we can readily find values similar to that already found for N_1. In fact we have

$$a^2 + \psi_1 = \frac{a^2 - c^2}{\mathrm{sn}^2\theta_1}, \quad b^2 + \psi_1 = (a^2 - c^2)\frac{\mathrm{dn}^2\theta_1}{\mathrm{sn}^2\theta_1}, \qquad c^2 + \psi_1 = (a^2 - c^2)\frac{\mathrm{cn}^2\theta_1}{\mathrm{sn}^2\theta_1},$$

giving

$$N_1 = (a^2 - c^2)^{\frac{1}{2}}\frac{\mathrm{dn}\,\theta_1\,\mathrm{cn}\,\theta_1}{\mathrm{sn}^3\theta_1}.$$

Also

$$a^2 + \psi_2 = (a^2 - c^2)\,\mathrm{dn}^2\,\theta_2, \quad b^2 + \psi_2 = (b^2 - c^2)\,\mathrm{cn}^2\,\theta_2, \quad c^2 + \psi_2 = -(b^2 - c^2)\,\mathrm{sn}^2\theta_2,$$

and

$$a^2 + \psi_3 = (a^2 - c^2)k^2\mathrm{sn}^2\,\theta_3,\ b^2 + \psi_3 = -(a^2 - c^2)\,k^2\,\mathrm{cn}^2\,\theta_3,\ c^2 + \psi_3 = -(a^2 - c^2)\,\mathrm{dn}^2\theta_3.$$

From these we get

$$\psi_1 - \psi_2 = (a^2 - c^2)\left\{\frac{1}{\mathrm{sn}^2\theta_1} - \mathrm{dn}^2\,\theta_2\right\},$$

$$\psi_1 - \psi_3 = (a^2 - c^2)\left\{\frac{1}{\mathrm{sn}^2\theta_1} - k^2\,\mathrm{sn}^2\,\theta_3\right\}.$$

Substituting in N_2 and N_3 the values now found for $a^2 + \psi_2$, etc., these become

$$N_2 = i\,(b^2 - c^2)\,\sqrt{a^2 - c^2}\,\mathrm{sn}\,\theta_2\,\mathrm{cn}\,\theta_2\,\mathrm{dn}\,\theta_2,$$

$$N_3 = (a^2 - b^2)\,\sqrt{a^2 - c^2}\,\mathrm{sn}\,\theta_3\,\mathrm{cn}\,\theta_3\,\mathrm{dn}\,\theta_3;$$

the above integral now becomes

$$= \frac{abc}{k'^2}\int_0\!\!\int_0 (\mathrm{dn}^2\,\theta_2 - k^2\sin^2\theta_3)\,d\theta_2 d\theta_3,$$

since

$$\frac{d}{d\theta}\,\mathrm{sn}\,\theta = \mathrm{cn}\,\theta\,\mathrm{dn}\,\theta,$$

$$\frac{d}{d\theta}\,\mathrm{dn}\,\theta = -\,k^2\,\mathrm{sn}\,\theta\,\mathrm{cn}\,\theta.$$

Now

$$\int_0 \mathrm{dn}^2\,\theta_2 d\theta_2 = \frac{E}{K}\,\theta_2 + Z\theta_2,$$

where $Z\theta_2$ is the periodic part of the integral, and

$$-\,k^2\!\int_0 \mathrm{sn}^2\,\theta_3 = Z\theta_3 + \left(\frac{E}{K} - 1\right)\theta_3.$$

Therefore, finally,

$$\iint \omega d\epsilon' = \frac{abc}{k'^2}\left\{\int_0 d\theta_3\left[\frac{E}{K} + Z\theta_2\right] + \int_0 d\theta_2\left[Z\theta_3 + \left(\frac{E}{K} - 1\right)\theta_3\right]\right\},$$

the integration to be extended over the entire surface of the ellipsoid. We will denote this integral by J, and then shall have for the surface potential

$$\Psi = \frac{J}{r}.$$

For the determination of the total pressure upon the body we have now

$$\int p_1 d\epsilon = -\int p_1 d\epsilon';$$

the right-hand member of this equation, which we will denote by $-X$, can thus be computed by merely integrating p_1 over the surface of the infinite sphere. Suppose, first, the case of very slow motions; then, as we are only concerned with the value of p at infinity, we neglect the squares and products of the velocities, also the terms containing V,

$$dp = 2\,a\mu \frac{d^2\Psi}{dx^2}\,dx,$$

$$p = 2\,a\mu \frac{d\Psi}{dx}.$$

Now (Lamb's Fluid Motion, page 217),

$$p_1 = lp_{xx} + mp_{xy} + np_{xz}, \text{ etc.},$$

where l, m, n are the direction-cosines of the normal to the bounding surface; in this case

$$= \frac{x}{r}, \frac{y}{r}, \frac{z}{r},$$

respectively. Also

$$p_{xx} = -p + 2\,\mu \frac{du}{dx},$$

$$p_{xy} = \mu\left(\frac{du}{dy} + \frac{dv}{dx}\right),$$

$$p_{xz} = \mu\left(\frac{du}{dz} + \frac{dw}{dx}\right), \text{ etc.}$$

The other relations are well known. We have now

$$\frac{du}{dx} = ax\frac{d^2\Psi}{dx^2},$$

$$\frac{du}{dy} + \frac{dv}{dx} = 2\,ax\frac{d^2\Psi}{dxdy},$$

$$\frac{du}{dz} + \frac{dw}{dx} = 2\,ax\frac{d^2\Psi}{dxdz}.$$

We have then

$$p_1 = 2\,a\mu\left\{\frac{d\Psi}{dx}\cdot\frac{x}{r} - \frac{x}{r}\cdot r\frac{d}{dr}\cdot\frac{d\Psi}{dx}\right\},$$

or
$$p_1 = 2\,a\mu\,\frac{x}{r}\left\{\frac{d\Psi}{dx} - r\,\frac{d}{dx}\,\frac{d\Psi}{dr}\right\}$$
$$= -\,6\,a\mu\,\frac{Jx^2}{r^4}.$$

Write $\frac{x}{r} = \cos \chi$, multiply by $2\,\pi r^2 \sin \chi d\chi$ and integrate from $\chi = 0$ to $\chi = 2\,\pi$;
$$\iint p_1 d\epsilon = -\,8\,\pi a\mu J.$$
Therefore we have
$$X = 8\,\pi a\mu J.$$

Introducing the found values of a and J, this is
$$X = \frac{4\,\pi\mu\lambda\,(a^2 - c^2)^{\frac{3}{2}}}{k'^2}\left\{\frac{\int d\theta_3\left[\frac{E}{K}\,\theta_2 + Z\theta_2\right] + \int d\theta_2\left[Z\theta_3 + \left(\frac{E}{K} - 1\right)\theta^3\right]}{a^2 \int_0 \mathrm{sn}^2\,\theta_1\,d\theta_1 - 2\,\pi\,(a^2 - c^2)\,K}\right\}$$
or
$$X = \frac{k^2}{k'^2}\,4\,\pi\mu\lambda\,(a^2 - c^2)^{\frac{3}{2}}\,\frac{\int d\theta_3\left[\frac{E}{K}\,\theta_2 + Z\theta_2\right] + \int d\theta_2\left[Z\theta_3 + \left(\frac{E}{K} - 1\right)\theta_3\right]}{a^2\left[\left(1 - \frac{E}{K}\right)\theta_1 - Z\theta_1\right] - 2\,\pi\,(a^2 - b^2)\,K}.$$

The determination of this quantity in the general case will be extremely difficult, probably impossible. It will first be necessary to find the pressure of p from either of the formulas, no external forces acting,
$$\frac{p}{\rho} = G - q - \left\{\frac{1}{2\,\pi}\iiint\left(\frac{dL}{dx'} + \frac{dM}{dy'} + \frac{dN}{dz'}\right)\frac{dx'dy'dz'}{r}\right\}$$
where
$$L = v\zeta - w\eta, \qquad M = -\,u\zeta, \qquad N = u\eta,$$
or from
$$\frac{p}{\rho} = G - q + \frac{1}{2\pi}\iint\frac{q'\Omega\,\sin\delta\,\sin\phi}{r}\,d\sigma + \frac{1}{2\pi}\iiint\frac{q'\Omega\,\sin\delta\,\sin\phi'}{r^2}\,dx'dy'dz'.$$
The stream and vortex lines do not cut constantly at right angles, so no simplification of this last is possible. A further investigation of this problem would require as a first step a thorough examination of the elliptic vortex rings formed in the fluid.*

We have for the angular velocities
$$\xi = 0,$$
$$\eta = -\,a\,\frac{d\Psi}{dz},$$
$$\zeta = a\,\frac{d\Psi}{dy}.$$

* On this point see an article "On Circular Vortex Rings" by C. V. Coates, Quarterly Journal, Vol. XVI. p. 170.

These differential coefficients are known to be given by

$$\frac{d\Psi}{dy} = 2\,y\,\frac{(a^2+\psi_1)\,(c^2+\psi_1)}{(\psi_1-\psi_2)\,(\psi_1-\psi_3)},$$

$$\frac{d\Psi}{dz} = 2\,z\,\frac{(a^2+\psi_1)\,(b^2+\psi_1)}{(\psi_1-\psi_2)\,(\psi_1-\psi_3)};$$

we have then for the differential equation of the vortex lines

$$\frac{ydy}{b^2+\psi_1} + \frac{zdz}{c^2+\psi_1} = 0;$$

for $\psi_1 = 0$, or all points on the surface of the ellipsoid, this gives

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \text{const.}$$

Introducing elliptic functions,

$$\frac{d\Psi}{dy} = \frac{2\sqrt{a^2-c^2}\,\mathrm{cn}^2\theta_1\,\mathrm{dn}\,\theta_1\,\mathrm{cn}\,\theta^2\,\mathrm{cn}\,\theta_3}{\mathrm{sn}\,\theta_1\,\{1-\mathrm{sn}^2\theta_1\,(\mathrm{dn}^2\theta_2+k^2\,\mathrm{sn}^2\theta_3)+k^2\,\mathrm{sn}^2\theta_1\,\mathrm{dn}^2\theta_2\,\mathrm{sn}^2\theta_3\}},$$

$$\frac{d\Psi}{dz} = \frac{2\sqrt{a^2-c^2}\,\mathrm{dn}^2\theta_1\,\mathrm{cn}\,\theta_1\,\mathrm{sn}\,\theta_2\,\mathrm{dn}\,\theta_3}{\mathrm{sn}\,\theta_1\,\{1-\mathrm{sn}^2\theta_1\,(\mathrm{dn}^2\theta_2+k^2\,\mathrm{sn}^2\theta_3)+k^2\,\mathrm{sn}^2\theta_1\,\mathrm{dn}^2\theta_2\,\mathrm{sn}^2\theta_3\}}.$$

WASHINGTON, D. C., Nov. 22, 1880.

On Binomial Congruences; comprising an Extension of Fermat's and Wilson's Theorems, and a Theorem of which both are Special Cases.

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THE complete theory of the binomial congruence $x^n \equiv D$ mod. k divides itself naturally into four parts. (1) The determination of n constitutes the theory of the periodicity of power residues;* (2) of x, the theory of the roots of the congruence; (3) of D, the theory of the number and the character of power residues; (4) of k, the theory of cyclotomic divisors, treated of in detail by Professor Sylvester in Vol. II., No. 4, of this Journal.

The theorems commonly known in the first three divisions of this subject have reference only to those numbers which are prime to the modulus. It is proposed in this article to so generalize certain fundamental theorems in (1), (2), and (3), including Fermat's and Wilson's, that they shall apply to all numbers, and to give a single theorem under which Fermat's and Wilson's theorems, thus extended, are both included as special cases.

§ 1. *Introductory Definitions and Notation.*

The number of numbers prime to and less than a given number, k, Professor Sylvester has named the *totient* of k, and, instead of the old symbol, $\phi(k)$, he uses $\tau(k)$ to designate it. In the following I shall speak of the number of numbers less than k, containing one and only one of its unequal prime factors, as the totient of k with respect to that prime factor. Thus, if $k = a^t b^r c^u$, where a, b, and c are prime numbers, I shall speak of the totient of k with respect to a, or the a-totient of k, and shall designate it by $\tau_a(k)$, in conformity with Professor Sylvester's notation. Likewise, the number of numbers less than k which contain a and b, but not c, I shall call the ab-totient of k, and write it $\tau_{ab}(k)$.

* *Power residues* is a term not used, I believe, but a needed translation of *Potenz-Reste*.

In general, representing by s the product of any given number of the unequal prime factors of a number, k, I shall speak of the s-totient of k, and denote it by $\tau_s(k)$. It is plain that, if $k = a^t b^u c^v$,

$$\tau_a(k) = a^{t-1}\tau(b^u c^v) = a^{t-1} b^{u-1} c^{v-1} (a-1)(c-1).$$

For there are $a^{t-1} b^u c^v$ numbers not greater than k which contain a, and, dividing these into a^{t-1} successive groups of $b^u c^v$ numbers each, we must reject from each group those numbers that contain either b or c, which leaves $a^{t-1}\tau(b^u c^v)$ numbers. In the same way it is plain that

$$\tau_{ab}(k) = a^{t-1} b^{u-1} \tau(c^v) = a^{t-1} b^{u-1} c^{v-1} (c-1).$$

We may write the different totients of $k = a^t b^u c^v$ in the following symmetrical manner, viz.:

$$\begin{aligned}
\tau_1(k) &= a^{t-1} b^{u-1} c^{v-1} & (a-1)\,(b-1)\,(c-1),\\
\tau_a(k) &= \quad\text{"} & (\ldots.)\,(b-1)\,(c-1),\\
\tau_b(k) &= \quad\text{"} & (a-1)\,(\ldots.)\,(c-1),\\
\tau_c(k) &= \quad\text{"} & (a-1)\,(b-1)\,(\ldots.),\\
\tau_{bc}(k) &= \quad\text{"} & (a-1)\,(\ldots.)\,(\ldots.),\\
\tau_{ca}(k) &= \quad\text{"} & (\ldots.)\,(b-1)\,(\ldots.),\\
\tau_{ab}(k) &= \quad\text{"} & (\ldots.)\,(\ldots.)\,(c-1),\\
\tau_{abc}(k) &= \quad\text{"} & (\ldots.)\,(\ldots.)\,(\ldots.).
\end{aligned}$$

The sum of these numbers is, of course, equal to k. This may readily be seen by noticing that the different terms of the expansion of

$$[\tau(a^t) + a^{t-1}]\,[\tau(b^u) + b^{u-1}]\,[\tau(c^v) + c^{v-1}]$$

are the above numbers, and this expression readily reduces to $a^t b^u c^v$. If $i =$ the number of unequal prime factors in any number, k, it is plain that the number of its different totients is 2^i.

It is convenient to have a name for those numbers whose enumeration makes up a totient of k. Professor Sylvester has called them, in the case of the ordinary totient, the *totitives* of k. Following this nomenclature, I shall be understood in speaking of the a-totitives, the ab-totitives, and, in general, of the s-totitives of k. X_s will conveniently denote *any s-totitive* of k. I shall use s throughout in the sense already defined, and shall use σ to denote the product of the same factors denoted by the s in the context, each factor, however, being

affected with the exponent which it has in k. Thus, if $k = a^t b^u \ldots. l^x \ldots. q^z$, and $s = ab \ldots. l$, then $\sigma = a^t b^u \ldots. l^x$. I shall use w frequently as $= ab \ldots. q$ = the product of all the unequal prime factors of k. I shall call a^t, b^u, etc., the *components* of k. Thus, we can define σ as the product of a certain number of the components of k, and s as the product of all the unequal prime factors of σ. If $s = 1$, then $\sigma = 1$.

§ 2. *On the Number and the Properties of the Roots of* $x^2 \equiv x$ *mod.* k.*

The solutions of $x^2 \equiv x$ mod. k have an important part in what follows, and will here be noticed somewhat in detail. This congruence breaks up into $x \equiv 0$ mod. σ and $x - 1 \equiv 0$ mod. $\frac{k}{\sigma}$, where σ and $\frac{k}{\sigma}$ are prime to each other, since x and $x - 1$ are necessarily prime to each other. We may write $x \equiv 0$ mod. σ as $x = \lambda\sigma$, whence the second congruence becomes $\lambda\sigma \equiv 1$ mod. $\frac{k}{\sigma}$. The last congruence gives one and only one value of λ less than $\frac{k}{\sigma}$, and, since $x = \lambda\sigma$, there is one and only one value of x corresponding to this value of λ which satisfies the congruence $x^2 \equiv x$ mod. k. The x obtained in this way is plainly an s-totitive of k. Designate it by R_s. It is evident, now, that there are twice as many solutions of this congruence as there are ways of breaking up k into two factors prime to each other, viz.: 2^i where i = the number of the components of k. Different separations give different solutions. For suppose $x = \lambda\sigma$, and $\lambda\sigma \equiv 1$ mod. $\frac{k}{\sigma}$ to have the same solution as $x = \lambda'\sigma'$ and $\lambda'\sigma' \equiv 1$ mod. $\frac{k}{\sigma'}$. Then $\lambda\sigma$ is divisible by both σ and σ', and $\lambda\sigma - 1$ is divisible by both $\frac{k}{\sigma}$ and $\frac{k}{\sigma'}$. But $\lambda\sigma$ and $\lambda\sigma - 1$, differing by unity, are necessarily prime to each other. This gives $\sigma\sigma'$ prime to $\frac{k}{\sigma} \cdot \frac{k}{\sigma'}$, which is impossible if σ and σ' are different factors of k. Hence we have

THEOREM I. *The congruence* $x^2 \equiv x$ *mod.* k *has* 2^i *different roots, one root belonging to each of the* 2^i *classes of the totitives of* k.

If $k = a^t b^u c^v$, the eight solutions of the congruence may be denoted by R_1, R_a, R_b, R_c, R_{bc}, R_{ca}, R_{ab}, R_{abc}, the subscript denoting in each case to what class of totitives the root belongs. R_1 always $= 1$, found by solving $x \equiv 0$ mod. 1 and $x \equiv 1$ mod. k. R_w always $= 0$, found by solving $x \equiv 0$ mod. k and $x \equiv 1$ mod. 1.

From $x^2 \equiv x$ mod. k we readily obtain $x^n \equiv x$ mod. k. The solutions above

* In Serret's *Cours d'Algèbre Supérieure*, § 292, the solution of $x^2 \equiv 1$ mod. k is discussed, a congruence which can be transformed into $x^2 \equiv x$ mod. k by substitution. I solve the latter congruence anew rather than transform results, both because its solution is simpler, and because the properties of its roots are more fundamental.

are, then, *repeating power residues* of the modulus k, and, in accordance with a suggestion from Professor Sylvester, are called the *repetents* of k.

I shall now prove some theorems in regard to the addition, subtraction, and multiplication of these repetents.

In all that follows I suppose s, s', s'', etc., to be prime to one another, unless otherwise stated.

THEOREM II. $R_s R_{s'} \equiv R_{ss'}$ *mod.* k.

We saw, in the course of the proof of Theorem I., that $R_s \equiv 0$ mod. σ, and $R_s \equiv 1$ mod. $\frac{k}{\sigma}$; that $R_{s'} \equiv 0$ mod. σ, and $R_{s'} \equiv 1$ mod. $\frac{k}{\sigma'}$; and that $R_{ss'} \equiv 0$ mod $\sigma\sigma'$ and $R_{ss'} \equiv 1$ mod. $\frac{k}{\sigma\sigma'}$. Whence $R_s R_{s'} \equiv 0 \equiv R_{ss'}$ mod. $\sigma\sigma'$, and $R_s R_{s'} \equiv 1 \equiv R_{ss'}$ mod. $\frac{k}{\sigma\sigma'}$. $\therefore R_s R_{s'} \equiv R_{ss'}$ mod. k. Q. E. D.

COROLLARY. $R_{ss'} R_{ss''} \equiv R_{ss's''}$ mod. k, since $R_s R_s \equiv R_s$ mod. k.

THEOREM III. $R_s + R_{s'} \equiv R_{ss'} + 1$ mod. k.

For $R_s - 1 \equiv 0$ mod. $\frac{k}{\sigma}$ and $R_{s'} - 1 \equiv 0$ mod. $\frac{k}{\sigma'}$, and by multiplication,

$$(R_s - 1)(R_{s'} - 1) \equiv 0 \equiv R_s R_{s'} - R_s - R_{s'} + 1 \text{ mod. } k.$$

$$\therefore R_s + R_{s'} \equiv R_{ss'} + 1 \text{ mod. } k.$$

COROLLARY 1. $R_s + R_{s'} \equiv 1$ mod. k when $ss' = w$, for then $R_{ss'} \equiv 0$ mod. k.

COROLLARY 2. $R_{ss'} + R_{ss''} \equiv R_{ss's''} + R_s$ mod. k. *Hence, the sum of any two of the repetents of* k *is congruous to the sum of any other two, provided the product of the subscripts is the same for each sum.*

COROLLARY 3. $(R_s - R_{s'})^2 \equiv 1 - R_{ss'}$ mod. k, and when $ss' = w$, then $R_{ss'} \equiv 0$, and we have $(R_s - R_{s'})^2 \equiv 1$ mod. k, obtained by squaring and reducing. $R_s - R_{s'}$ is, then, a root of $x^2 \equiv 1$ mod. k, in conformity with the relation existing between the algebraic roots of $x^2 = x$ and $x^2 = 1$. In general, $R_s - R_{s'}$ is a root of $X^2 \equiv R_{\overline{ss'}}$ mod. k, where $\overline{ss'} = \frac{w}{ss'}$.

THEOREM IV. $R_s - R_{s'} \equiv R_{ms} - R_{ms'}$ mod. k, m being any product of the unequal prime factors of k which is prime to s and s', and the relation holds whether s and s' are prime to each other or not. In words, *The residue mod.* k, *of the difference of any two of the repetents of* k *remains constant if their subscripts be both multiplied by any prime factor or factors not contained in either, or both divided by any prime factor or factors common to both.* For, by Theorem III., Corollary 2,

$$R_s + R_{ms'} \equiv R_{s'} + R_{ms} \text{ mod. } k.$$

The most general theorem of summation may be stated as follows, viz.:—

THEOREM V. *The modulus being* k, *the sum of a given number of repetents is congruous to the sum of the same number of any other repetents, provided only that the product of all the subscripts is the same in each sum.*

For any sum of n repetents may plainly be transformed, by Theorem III., Corollary 2, in such a way that the first term will have for its subscript only those letters which occur in all the n terms. If there be no letters which occur in all the n terms, the first term may be transformed into R_1. The second term may be made to have for its subscript, in addition to those contained in the first term, only those letters which occur in the remaining $(n-1)$ terms, and so on, till the last term whose subscript contains all the different letters of the preceding terms, together with any which may occur in only one of the terms. Thus,

$$R_{cdf} + R_{cde} + R_f + R_{cf} + R_{ef} + R_{bcf} + R_{def} + R_{abc} \equiv R_1 + R_1 + R_f + R_{cf} + R_{cf} \\ + R_{cf} + R_{bcdef} + R_{abcdef} \text{ mod. } a^t b^u c^v d^w e^x f^y$$

by such a transformation. The number of terms remains the same and the product of the subscripts remains the same. Evidently, every sum of terms which fulfils the given conditions can be reduced to the same sum. Hence the theorem is proved.

COROLLARY 1. $R_s + R_{s'} + \ldots + R_{s^{[n]}} \equiv R_{ss's''\ldots s^{[n]}} + nR_1$ mod. k.

COROLLARY 2. $\Sigma R_{ss's''\ldots s^{[r]}} \equiv C_r^n R_{ss's''\ldots s^{[n]}} + C_{r+1}^n$ mod. k, where C_r^n denote the r^{th} coefficient of the n^{th} power of a binomial. The first member of the congruence denotes the sum of those repetents whose subscripts are the C_{r+1}^{n+1} combinations of the $(n+1)$ S's taken $(r+1)$ at a time.

COROLLARY 3. If $k = a^t b^u \ldots q^z$, we have

$$R_1 + \Sigma R_a + \Sigma R_{ab} + \ldots + \Sigma R_{ab\ldots p} + R_{ab\ldots q} \equiv 2^{i-1} \text{ mod. } k,$$

where $i =$ the number of the components of k. And also

$$R_1 - \Sigma R_a + \Sigma R_{ab} - \Sigma R_{abc} + \ldots (-)^i R_{ab\ldots q} \equiv 0 \text{ mod. } k.$$

If we write $\bar{s}$ to denote briefly $\frac{w}{s}$, where w is the product of all the unequal prime factors of the modulus, k, we shall have a convenient notation for expressing certain formulæ which will be most useful further on. As before, let s, s', s'', etc., be prime to one another, but of course $\bar{s}, \bar{s'}, \bar{s''}$ will not be so. It is to be remembered that R_s contains only those factors of k which are found in s, whereas $R_{\bar{s}}$ contains only those *not* found in s, so that the dash over s has the significance of a logical negative.

$$\text{(A)} \qquad R_{\bar{s}} R_{\bar{s}'} \equiv 0 \text{ mod. } k.$$

$$\text{(B)} \qquad R_{\bar{s}} + R_{\bar{s}'} \equiv R_{\overline{ss'}} \text{ mod. } k.$$

$$\text{(C)} \qquad \Sigma_0^n R_{\bar{s}} \equiv R_{\overline{ss's''\ldots\, s^{[n]}}} \text{ mod. } k.$$

$$\text{(D)} \qquad \Sigma R_{\overline{ss's''\ldots\, s^{[r]}}} \equiv C_r^n R_{\overline{ss's''\ldots\, s^{[n]}}} \text{ mod. } k.$$

$$\text{(E)} \qquad R_{\bar{s}} - R_{\bar{s}'} \equiv -(R_s - R_{s'}) \text{ mod. } k.$$

(A) follows, since $\frac{w}{s} \cdot \frac{w}{s'}$ contains w. (B) is only Corollary 2, Theorem III., in another form. (C) follows directly from (B), and (D) from (C).

THEOREM VI. If $X \equiv \alpha R_{\bar{s}} + \beta R_{\bar{s}'} + \ldots. + \lambda R_{\overline{s^{[n]}}}$ mod. k, and $X' \equiv \alpha' R_{\bar{s}} + \beta R_{\bar{s}'} + \cdots + \lambda' R_{\overline{s^{[n]}}}$, and so on, for X'', X''', etc., where $\alpha, \beta, \ldots. \alpha', \beta', \ldots.$ are any integers, then

$$(XX'X''\ldots.) \equiv (\alpha\alpha'\alpha''\ldots.) R_{\bar{s}} + (\beta\beta'\beta''\ldots.) R_{\bar{s}'} + \ldots. + (\lambda\lambda'\lambda'') R_{\overline{s^{[n]}}} \text{ mod. } k.$$

For since $R_{\bar{s}} R_{\bar{s}'} \equiv 0$, all cross multiplication will give terms congruous to zero.

COROLLARY. If $X \equiv \alpha R_{\bar{s}} + \beta R_{\bar{s}'} + \ldots. + \lambda R_{\overline{s^{[n]}}}$ mod. k, as before, then $X^m \equiv \alpha^m R_{\bar{s}} + \beta^m R_{\bar{s}'} + \ldots. + \lambda^m R_{\overline{s^{[n]}}}$ mod. k.

THEOREM VII.* *If* $\mathrm{X} \equiv \alpha \mathrm{R}_{\bar{\mathrm{a}}} + \beta \mathrm{R}_{\bar{\mathrm{b}}} + \ldots. + \chi \mathrm{R}_{\bar{\mathrm{q}}}$ *mod.* $\mathrm{k} = \mathrm{a}^t \mathrm{b}^u \ldots. \mathrm{q}^z$, *then* $\mathrm{X} \equiv \alpha$ *mod.* a^t, $\mathrm{X} \equiv \beta$ *mod.* b^u, *etc., and, vice versa, if* $\mathrm{X} \equiv \alpha$ *mod.* a^t, $\mathrm{X} \equiv \beta$ *mod.* $\mathrm{b}^u, \ldots.$ *and* $\mathrm{X} \equiv \chi$ *mod.* q^z, *then* $\mathrm{X} \equiv \alpha \mathrm{R}_{\bar{\mathrm{a}}} + \beta \mathrm{R}_{\bar{\mathrm{b}}} + \ldots. + \chi \mathrm{R}_{\bar{\mathrm{q}}}$ *mod.* $\mathrm{k} = \mathrm{a}^t \mathrm{b}^u \ldots. \mathrm{q}^z$. These results follow from the definition of the repetents, $R_{\bar{a}}$, $R_{\bar{b}}$, etc.

THEOREM VIII. *If* $\mathrm{AR}_{\bar{\mathrm{a}}} + \mathrm{BR}_{\bar{\mathrm{b}}} + \ldots. + \mathrm{QR}_{\bar{\mathrm{q}}} \equiv \mathrm{A'R}_{\bar{\mathrm{a}}} + \mathrm{B'R}_{\bar{\mathrm{b}}} + \ldots. + \mathrm{Q'R}_{\bar{\mathrm{q}}}$ *mod.* $\mathrm{k} = \mathrm{a}^t \mathrm{b}^{t'} \ldots. \mathrm{q}^{t''}$, *then* $\mathrm{A} \equiv \mathrm{A}'$, $\mathrm{B} \equiv \mathrm{B}', \ldots.,$ *mod.* k. This follows from the definition of $R_{\bar{a}}$, $R_{\bar{b}}$, etc. [See Dirichlet's *Zahlentheorie*, § 25.]

THEOREM IX. *If* $\mathrm{f(X)} \equiv 0$ *mod.* $\mathrm{k} = \mathrm{a}^t \mathrm{b}^{t'} \ldots. \mathrm{q}^{t''}$ *be transformed by the substitution* $\mathrm{X} \equiv \mathrm{uR}_{\bar{\mathrm{a}}} + \mathrm{vR}_{\bar{\mathrm{b}}} + \ldots. + \mathrm{yR}_{\bar{\mathrm{q}}}$, *it becomes* $\mathrm{f(u)R}_{\bar{\mathrm{a}}} + \mathrm{f(v)R}_{\bar{\mathrm{b}}} + \ldots. + \mathrm{f(y)R}_{\bar{\mathrm{q}}} \equiv 0$ *mod.* k, *which involves* $\mathrm{f(u)} \equiv 0$ *mod.* a^t, $\mathrm{f(v)} \equiv 0$ *mod.* $\mathrm{b}^{t'}$, *etc.* This is obvious from Theorem VII.

THEOREM X. *If* N_s *be any number which contains* σ, *then* $\mathrm{R}_s \mathrm{N}_s \equiv \mathrm{N}_s$ *mod.* k. For $R_s \equiv 1$ mod. $\frac{k}{\sigma}$.

I give below the repetents of the modulus 210, for the convenience of any reader who may wish to illustrate the preceding theorems by examples.

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* In Dirichlet's *Zahlentheorie*, § 25, this solution of a system of linear congruences is given. Aa', Bb', ... are used instead of $R_{\bar{a}}$, $R_{\bar{b}}$, ..., where $A = \frac{k}{a^t}$, $B = \frac{k}{b^u}$, ..., and a', b', ..., are to be determined in such a way that $Aa' \equiv 1$ mod. a^t, $Bb' \equiv 1$ mod. b', etc., etc. These conditions are the same as those which, as we have seen, the repetents $R_{\bar{a}}$, $R_{\bar{b}}$, ... fulfil.

Mod. $210 = 2.3.5.7$.

$R_1 = 1$	$R_2 = 106$ $R_3 = 141$ $R_5 = 85$ $R_7 = 91$	$R_{2.3} = 36$ $R_{2.5} = 190$ $R_{2.7} = 196$ $R_{3.5} = 15$ $R_{3.7} = 21$ $R_{5.7} = 175$	$R_{3.5.7} = 105$ $R_{5.7.2} = 70$ $R_{7.2.3} = 126$ $R_{2.3.5} = 120$	$R_{2.3.5.7} = 0$

§ 3. *Fermat's and Wilson's Theorems.*

On examination of a table of power residues for a given modulus, it will be seen that all the repetents of the modulus play the same part in the table as unity. Thus, in the following table,

Mod. 15.

Natural Numbers	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Quad. Residues .	0	1	4	9	1	10	6	4	4	6	10	1	9	4	1
Cubic Residues .	0	1	8	12	4	5	6	13	2	9	10	11	3	7	14
Biquad. Residues	0	1	1	6	1	10	6	1	1	6	10	1	6	1	1

the repetents 0, 6, and 10 have a part similar to that of unity, which is itself one of the four repetents of 15. These numbers which we have called repetents are, in fact, a kind of residual units. Their fundamental property, $R_s^n \equiv R_s$ mod. k, shows their analogy to ordinary unity, and Theorem II., which gives the product of two repetents congruous to a certain third one, shows their analogy to the multiple units of quaternions, where we have $ij = k$. Their analogy to the double units of imaginary or complex quantities is seen in Theorems VIII. and IX.

It will now seem almost obvious that Fermat's, Wilson's, and other theorems in power residues ought to be extensible in such a way that all these repetents, or residual units, shall have the same part as unity has in the present forms of the theorems. It will now be shown that such is the case.

Extension of Fermat's Theorem for Composite Moduli. $X_s^{\tau_s(k)} \equiv R_s$ mod. k. For a proof I generalize one of the proofs of the ordinary theorem. Let $\alpha, \beta, \gamma, \ldots\ldots \omega$ be the $\tau_s(k)$ s-totitives of k. Multiply each member of the group by any one, as ρ. The series then becomes $\alpha\rho, \beta\rho, \gamma\rho, \ldots\ldots \omega\rho$. No two members of this latter series are congruous mod. k, for $(\theta - \delta)\rho$ cannot contain k. The

numbers are still s-totitives of k, for no new factor has been introduced by the multiplication. The numbers of the second series are, therefore, taken in some order, congruous respectively to the numbers of the first series, and we have $(\alpha\beta \ldots \omega)\,\rho^{\tau_s(k)} \equiv \alpha\beta \ldots \omega \bmod. k$, or $(\alpha\beta \ldots \omega)\,(\rho^{\tau_s(k)} - 1) \equiv 0 \bmod. k$. $\therefore \rho^{\tau_s(k)} \equiv 1 \bmod. \frac{k}{\sigma}$. But $R_s \equiv 1 \bmod. \frac{k}{\sigma}$ [Theorem I.], $\therefore \rho^{\tau_s(k)} \equiv R_s \bmod. \frac{k}{\sigma}$. Both sides of this last congruence evidently contain σ, for $R_s = \lambda\sigma$ [Theorem I.], and the unequal prime factors in ρ evidently cannot have smaller exponents in $\rho^{\tau_s(k)}$ than they have in σ. $\therefore \rho^{\tau_s(k)} \equiv R_s \bmod. k$, or, writing X_s for ρ,

$$X_s^{\tau_s(k)} \equiv R_s \bmod. k. \quad \text{Q. E. D.}$$

For $s = 1$, we have $R_s = 1$, and $x^{\tau(k)} \equiv 1 \bmod. k$, — the ordinary form of the Fermatian theorem for composite moduli, in which x is supposed prime to k. This is now seen to be only one of the 2^i different cases of the extended theorem.

I now give another proof dependent upon the ordinary theorem.

Let $k = a^t b^u \ldots l^x m^y \ldots q^z$, and $s = ab \ldots l$. If $X_s \equiv a' \bmod. a^t$, $X_s \equiv b' \bmod. b^u, \ldots X_s \equiv q' \bmod. q^z$, then $X_s \equiv a'R_{\bar a} + b'R_{\bar b} + \ldots + q'R_{\bar q} \bmod. k$ [§ 2, Theorem VII.].

$$\therefore X^n \equiv a'^n R_{\bar a} + b'^n R_{\bar b} + \ldots + q'^n R_{\bar q} \bmod. k \text{ [§ 2, Theorem VI.]};$$

$$\therefore \text{If } n = \tau_s(k),\ X^{\tau_s(k)} \equiv R_{\bar m} + \ldots + R_{\bar q} \equiv R_{\overline{m \ldots q}} = R_s \bmod. k.$$

For, from the linear congruences, a' contains a, b' contains $b, \ldots l'$ contains l, m' is prime to $m^y, \ldots$ and q' prime to q^z. Hence $a'^{\tau_s(k)} R_{\bar a} \equiv 0 \bmod. k, \ldots l'^{\tau_s(k)} R_{\bar l} \equiv 0 \bmod. k$, $m'^{\tau_s(k)} \equiv 1 \bmod. m^y, \ldots$ and $q'^{\tau_s(k)} \equiv 1 \bmod. q^z$, $\tau_s(k)$ containing $\tau(m^y), \ldots$, and $\tau(q^z)$. We have $R_{\bar m} \ldots R_{\bar q} \equiv R_{\overline{m \ldots q}} \bmod. k$, by § 2, Formula (C).

This method of proof especially brings out the analogy between this theorem and De Moivre's theorem concerning the n^{th} roots of unity.

Examples. If $k = 210 = 2.3.5.7$ [see table in last section], $R_{2.3} = 36$, and $X_{2.3}^{24} \equiv 36 \bmod. 210$, where $X_{2.3}$ is any 2.3-totitive of 210, i. e. any number which contains both 2 and 3, but *not* 5 or 7. So $X_{2.3.5}^{6} \equiv 120 \bmod. 210$, where $X_{2.3.5}$ is any number that contains 2, 3, and 5, but not 7, etc.

Whatever has been proved, Theorems II. to X. in regard to R_s, may now be stated in terms of $X_s^{\tau_s(k)}$. For instance, Corollary 3, Theorem VI., becomes, since $\tau(k)$ is a multiple of $\tau_s(k)$,

$$X_1^{\tau(k)} + \Sigma X_a^{\tau(k)} + \Sigma X_{ab}^{\tau(k)} + \ldots + X_{ab \ldots q}^{\tau(k)} \equiv 2^{i-1} \bmod. k.$$

The original theorem of Fermat, for prime numbers, may be regarded as a special case of this last congruence, for when k is prime, $2^{i-1} = 1$, and we have $X_1^{\tau(k)} \equiv 1 \bmod. k$.

If we designate by $\Pi_s(k)$ the product of the $\tau_s(k)$ s-totitives of k, we may state

An Extension of Wilson's Theorem for Composite Moduli. $\Pi_s(k) \equiv R_s$ *mod.* k, *except when* $\frac{k}{\sigma}$ *is a power of an odd prime number, double such a power, or* $= 4$, *and* $\frac{\sigma}{s}$ *is at the same time an odd number, in all of which cases* $\Pi_s(k) \equiv - R_s$ *mod.* k.

Proof. Since $\tau_s(k) = \frac{\sigma}{s}\,\tau\left(\frac{k}{\sigma}\right)$, we may divide the s-totitives of k into $\frac{\sigma}{s}$ groups of $\tau\left(\frac{k}{\sigma}\right)$ each, and may write the theorem to be proved as follows: —

$$s^{\tau_s(k)}\left[\Pi_1\left(\frac{k}{\sigma}\right)\right]\left[\Pi_1\left(\frac{k}{\sigma}\right),\text{ each} + \frac{k}{\sigma}\right]\left[\Pi_1\left(\frac{k}{\sigma}\right),\text{ each} + \frac{2k}{\sigma}\right]\left[\ldots\right]$$
$$\left[\Pi_1\left(\frac{k}{\sigma}\right),\text{ each} + \left(\frac{\sigma}{s} - 1\right)\frac{k}{\sigma}\right] \equiv \pm R_s \bmod. k.$$

We have here $\tau_s(k)$ numbers less than k, and no two of them are congruous; for suppose $ls + \lambda\frac{ks}{\sigma} \equiv ms + \mu\frac{ks}{\sigma} \bmod. k$, where l and m are prime totitives of $\frac{k}{\sigma}$. This may be written $(l - m)\,s \equiv (\mu - \lambda)\frac{ks}{\sigma} \bmod. k$, which is impossible, since the right member is divisible by $\frac{k}{\sigma}$ and the left is prime to $\frac{k}{\sigma}$.

R_s contains σ by Theorem I., and $\tau_s(k)$ is plainly in no case smaller than any exponent in σ. Therefore both sides contain σ whatever may be the sign of R_s. If $\frac{k}{\sigma}$ be a power of an odd prime number, double such a power, or $= 4$, each parenthesis is congruous to $-1 \bmod. \frac{k}{\sigma}$. If, further, $\frac{\sigma}{s}$ be at the same time an odd number, the product of all the parentheses is congruous to $-1 \bmod. \frac{k}{\sigma}$. In all other cases the product of the parentheses is congruous to $+1 \bmod. \frac{k}{\sigma}$. The congruence to be proved reduces, then, to $\pm s^{\tau_s(k)} \equiv \pm 1 \bmod. \frac{k}{\sigma}$, and, if 1 be taken with the same sign as the left-hand member, this is a valid congruence by the Fermatian theorem. Hence, the original congruence being valid for both mod. σ and mod. $\frac{k}{\sigma}$, it is valid for mod. k. Q. E. D.

Another proof of this theorem is as follows: Let $k = a^t b^u \ldots\ldots l^x m^y \ldots\ldots q^z$, and $s = ab \ldots\ldots l$. Then if $X'_s \equiv a' \bmod. a^t \ldots\ldots X'_s \equiv q' \bmod. q^z$, and $X''_s \equiv a''$ mod. $a^t \ldots\ldots X''_s \equiv q'' \bmod. q^z$, and so on for X''', etc.;

$$X'_s \equiv a'R_{\bar a} + b'R_{\bar b} + \ldots\ldots + q'R_{\bar q} \bmod. k,$$
$$X''_s \equiv a''R_{\bar a} + b''R_{\bar b} + \ldots\ldots + q''R_{\bar q} \quad \text{"} \quad \text{"}$$

and so on for X_s''', etc., all the $\tau_s(k)$ s-totitives of k being obtained in this way, since there are a^{t-1} terms, a', a'', etc., b^{u-1} terms, b', b'', etc., l^{x-1} terms, l', l'', etc., $\tau(m^y)$ terms, m', m'', etc., and $\tau(q^z)$ terms, q', q'', ..., and the product of these numbers $= a^{t-1}b^{u-1}\dots l^{x-1}\tau(m^y)\dots\tau(q^z) = \tau_s(k)$. Then we have

$$\Pi_s(k) \equiv \left(m'm''\dots m^{[\tau(m^y)]}\right)^{\frac{\tau_s(k)}{\tau(m^y)}} R_{\bar{m}} + \dots + \left(q'q''\dots q^{[\tau(q^z)]}\right)^{\frac{\tau_s(k)}{\tau(q^z)}} R_{\bar{q}}$$

$$\therefore \Pi_s(k) \equiv \left(-1\right)^{\frac{\tau_s(k)}{\tau(m^y)}} R_{\bar{m}} + \dots + \left(-1\right)^{\frac{\tau_s(k)}{\tau(q^z)}} R_{\bar{q}} \text{ mod. } k.$$

$$\frac{\tau^s(k)}{\tau(m^y)} = \frac{\sigma}{s}\,\tau\left(\frac{k}{m^y\sigma}\right), \dots \text{ and } \frac{\tau_s(k)}{\tau(q^z)} = \frac{\sigma}{s}\,\tau\left(\frac{k}{q^z\sigma}\right),$$

which are even numbers, except when $\frac{k}{\sigma}$ is a power of an odd prime number, double such a power, or $= 4$, and $\frac{\sigma}{s}$ is at the same time an odd number. Hence, outside of the excepted cases, we have

$$\Pi_s(k) \equiv R_{\bar{m}} + \dots + R_{\bar{q}} \equiv R_{\overline{m\dots q}} = R_s \text{ mod. } k,$$

[see § 2, Formula (C)]. For $\frac{\sigma}{s}$ odd, and $\frac{k}{\sigma} = q^z$, where q is an odd prime number, or where $q^z = 4$, we have $\Pi_s(k) \equiv -R_{\bar{q}} = -R_s$ mod. k. For $\frac{\sigma}{s}$ odd, and $\frac{k}{\sigma} = 2\,q^z$, where q is an odd prime number, we have

$$\Pi_s(k) \equiv \left(-1\right)^{\frac{\sigma}{s}\tau\left(\frac{k}{2\sigma}\right)} R_{\bar{2}} + \left(-1\right)^{\frac{\sigma}{s}\tau\left(\frac{k}{q^z\sigma}\right)} R_{\bar{q}} \text{ mod. } k.$$

$\therefore \Pi_s(k) \equiv + R_{\bar{2}} - R_{\bar{q}}$ mod. k. But when $k =$ twice an odd number, as in this case, we plainly have $R_{\bar{2}} = \frac{1}{2}k$, and therefore $R_{\bar{2}} \equiv -R_{\bar{2}}$ mod. k; so that $\Pi_s(k) \equiv -R_{\bar{2}} - R_{\bar{q}} \equiv -R_{\overline{2q}} = -R_s$ mod. k, according to § 2, Formula (C). Q. E. D.

Examples. If $k = 60 = 2^2 . 3 . 5$, then $R_{2.3} = 36$, $R_{2.5} = 40$, and $R_{3.5} = 45$. We have, therefore,

$\Pi_{2.3}(60) \equiv$ 36 mod. 60, i. e. $6 . 12 . 18 . 24 . 36 . 42 . 48 . 54 \equiv 36$,
$\Pi_{2.5}(60) \equiv$ 40 " " " $10 . 20 . 40 . 50 \equiv 40$,
$\Pi_{3.5}(60) \equiv -45$ " " " $15 . 45 \equiv -45$,

the sign of the right-hand member in the last congruence being negative, since $\frac{k}{\sigma} = \frac{60}{3.5} = 4$, and $\frac{\sigma}{s} = \frac{15}{3.5} = 1$.

For $s = 1$, we have, of course, $\Pi_1(60) \equiv 1$ mod. 60.

We may now state Theorems II. to X. in terms of $\Pi_s(k)$, care being taken in regard to signs. For instance, Theorem II. now becomes

$$\Pi_s(k)\,\Pi_{s'}(k) \equiv \pm\ \Pi_{ss'}(k) \text{ mod. } k,$$

the sign of the right-hand member being determined according to the previous theorem.

§ 4. *The Periodicity of Power Residues.*

THEOREM I. *Every power residue of an* s-*totitive of* k, X_s, *is also an* s-*totitive of* k. For if D be an n^{th} power residue of X_s mod. k, we have $X_s^n - D = \lambda k$, from which we see that D contains s and is prime to $\frac{k}{\sigma}$, and is therefore an s-totitive of k, for, otherwise, we should have in each case the difference of two integers equal to a fraction.

THEOREM II. If $k = a^t b^u \ldots . l^y \ldots . q^z$, $\sigma = a^t b^u \ldots . l^y$, $X_s = \lambda a^{t'} b^{u'} \ldots . l^{y'}$ where λ is prime to k, and $v - v'$ be the greatest difference obtained by subtracting each of the exponents $t', u', \ldots . y'$, from $t, u, \ldots . y$, respectively, and $\phi =$ the number of integers in $\frac{v + v' - 1}{v'}$, then

The ϕ^{th} *and all higher power residues of* X_s *mod.* k *contain* σ, *and no power residue of less degree than* ϕ *contains* σ.

Since a divisor of two numbers divides their remainder after division, we have only to consider what value of n will render X_s^n divisible by σ, i. e. by $a^t b^u \ldots . l^y$; then, since $v'\phi =$ or $> v$, and $v'(\phi - 1) < v$, the theorem is proved.

THEOREM III. *The power residues mod.* k *of any number* X_s *occur periodically as the degree of the power increases; and the exponent of the period is a divisor of* $\tau_s(k)$.

No power residue (except R_s) can occur twice as the degree increases, unless R_s intervenes; for if $X_s^n \equiv X_s^{n+h}$ mod. k without any intermediate power congruous to R_s, it follows at once, by multiplying both sides of the congruence repeatedly by X_s, that no power residue is congruous to R_s, which is not true, for $X_s^{\tau_s(k)} \equiv R_s$ mod. k by the extended Fermatian Theorem.

Let X_s^δ be the lowest power of X_s which is congruous to R_s. We have, then, $X_s^\delta \equiv R_s$ mod. k, and, by squaring, $X_s^{2\delta} \equiv R_s$ mod. k, whence $X_s^{\delta+h} \equiv X_s^{2\delta+h}$ mod. k, h being any positive integer. This shows that from the δ^{th} degree upwards the power residues of X_s mod. k recur in the same order in periods of δ numbers each. But, according to the preceding theorem, the first $(\phi - 1)$ residues of the first period are different from the corresponding residues in the higher periods. The remaining $\delta - \phi + 1$ residues of the first period are the same as the corresponding residues in the second, third, and higher periods;

for, by multiplying together, member by member, $X_s^{\phi+h} \equiv X_s^{\phi+h}$ mod. k and $X_s^{\delta} \equiv R_s$ mod. k, we obtain $X_s^{\delta+\phi+h} \equiv X_s^{\phi+h}$ mod. k, since $R_s X_s^{\phi+h} \equiv X_s^{\phi+h}$ mod. k by Theorem X. in § 2.

When $\phi = 1$, i. e. when all the exponents of the unequal prime factors common to X_s and k are as great in X_s as in k, then the first period of the power residues of X_s is the same as the higher periods.

It is evident now that δ is a divisor of $\tau_s(k)$, for R_s occurs only at the end of each period of residues, and $X_s^{\tau_s(k)} \equiv R_s$ mod. k.

THEOREM IV. *The numbers given by the formula* $X_s + \frac{sk}{\sigma}y$, *where* $y =$ *any integer, have the same power residues in the same order beyond the* $(t-1)^{th}$ *degree,* t *being the greatest exponent in* σ. *There are* $\frac{\sigma}{s}$ *such numbers less than* k.

Proof. If X_s be any s-totitive of k, then all the s-totitives of k are included among the residues of $X_s + s\lambda$, where λ is any integer from 0 to $\frac{k}{s} - 1$. Let us consider for what values of λ, if for any, $X_s^n \equiv (X_s + s\lambda)^n$ mod. k, for all values of $n =$ or $> t$. Transposing and taking out the factor s^n, we have $s^n\{(Q+\lambda)^n - Q^n\} \equiv 0$ mod. k, where Q denotes $\frac{X_s}{s}$. Therefore, if $n =$ or $> t$, the largest exponent in σ, we have $(Q+\lambda)^n - Q^n \equiv 0$ mod. $\frac{k}{\sigma}$. Since λ is a factor of the left-hand member, the congruence will be satisfied by making $\lambda = \frac{k}{\sigma}y$, where y is any integer; and it may be shown that the congruence will not hold if λ is prime to any prime factor of $\frac{k}{\sigma}$, say p, for then $(Q+\lambda)^n \equiv Q^n$ mod. p, which is not true for *all* values of $n =$ or $> t$, $\frac{k}{\sigma}$ being prime to X_s, and p therefore prime to Q.

Substituting then $\frac{k}{\sigma}y$ for λ, we have $X^n \equiv \left(X_s + \frac{sk}{\sigma}y\right)^n$ mod. k. Q. E. D.

There are plainly $\frac{\sigma}{s}$ numbers less than k which have the same power residues in the same order beyond the $(t-1)^{\text{th}}$ power, for, giving y all values from 0 to $\frac{\sigma}{s} - 1$ in the formula, $X_s + \frac{sk}{\sigma}y$, we get $\frac{\sigma}{s}$ different numbers whose residues mod. k are all different.

Those numbers obtained by giving to y values prime to $\frac{\sigma}{s}$ contain the prime factors of s only to the first degree, and therefore the value of ϕ (see Theorem II.) for these numbers is t, the greatest exponent in σ. Hence for $n < t$ there will be a disagreement among the power residues of the numbers given by the formula $X_s + \frac{sk}{\sigma}y$.

Example. If $k = 360 = 2^3 . 3^2 . 5$, the four numbers less than k given by the formula $2 + \frac{360}{2^2} y$ all have the same power residues beyond the second power. Also the numbers, $14 + \frac{360}{2^2} y$. Also the numbers, $42 + \frac{360}{2^2 . 3} y$, and there are $\frac{2^2 . 3^2}{2 . 3} = 12$ of these less than k.

Since $\tau_s(k) = \frac{\sigma}{s} \tau_1 \left(\frac{k}{\sigma}\right)$, we see that the s-totitives of k have only $\tau_1 \left(\frac{k}{\sigma}\right)$ different periods of power residues. If $s = 1$, then $\sigma = 1$, and $X_s + \frac{sk}{\sigma} y$ gives only one number less than k, viz. X_1, and there are in this case $\tau_1(k)$ different periods of power residues.

§ 5. *The Number of Roots of* $X^n - R_s \equiv 0$ *mod.* k.

Let $k = a^t b^{t'} \dots . l^{t''} m^{t'''} \dots . q^{t^{iv}}$, and $s = ab \dots . l$. Then $\sigma = a^t b^{t'} \dots . l^{t''}$. If $\theta =$ the number of integers in $\frac{t + n - 1}{n}$, $X^n - R_s \equiv 0$ mod a^t has $a^{t-\theta}$ roots, for, since $n\theta =$ or $> t$, and $n(\theta - 1) < t$, the n^{th} powers of all those numbers (and of no others) which contain a^θ will contain a^t, i. e. all such numbers will satisfy $X^n - R_s \equiv 0$ mod. a^t, and there are $a^{t-\theta}$ of such numbers less than a^t. In the same way $X^n - R_s \equiv 0$ mod. $b^{t'}$ has $b^{t'-\theta'}$ roots, etc. Hence $X^n - R_s \equiv 0$ mod. σ has $a^{t-\theta} b^{t'-\theta'} \dots . l^{t''-\theta''}$ roots [Serret, *Alg. Sup.* § 325].

Since $R_s \equiv 1$ mod. $\frac{k}{\sigma}$, $X^n - R_s \equiv 0$ mod. $m^{t'''}$ has μ roots, where μ is the g. c. d. of n and $\tau(m^{t'''})$ [Serret, *Alg. Sup.* § 322], and χ roots mod. $q^{t^{iv}}$, where χ is the g. c. d. of n and $\tau(q^{t^{iv}})$. Hence $X^n - R_s \equiv 0$ mod. $\frac{k}{\sigma}$ has $\mu \dots . \chi$ roots. In case one of the prime factors of $\frac{k}{\sigma}$, as m, $= 2$, we have, when n is even and $m^{t^{iv}} > 4$, $\mu =$ double the g. c. d. of n and $\frac{1}{2} \tau(m^{t^{iv}})$ [Serret, *Alg. Sup.* § 325].

Hence we have that $X^n - R_s \equiv 0$ mod. k has $a^{t-\theta} b^{t'-\theta'} \dots . l^{t''-\theta''} \mu \dots . \chi$ roots.

If $n =$ or $>$ than any of the exponents in σ, then the number of roots becomes $a^{t-1} b^{t'-1} \dots . l^{t''-1} \mu \dots . \chi$.

If we denote any repetent of k by R without a subscript, then the whole number of roots of $X^n - R \equiv 0$ mod. k is

$$(\alpha + a^{t-\theta})(\beta + b^{t'-\theta'}) \dots . (\chi + q^{t^{iv}-\theta^{iv}}),$$

for the different terms of this expansion are, by what has just been given, the numbers of roots for the different values of the repetents $R_1, R_a, R_b, \dots . R_{ab}$,

$R_{bc}, \ldots . R_{abc}, \ldots . R_{ab \ldots . q}$, there being the same number of terms as there are values of R, viz. 2^i, where $i =$ the number of the components of k.

Example. If $k = 72 = 2^3 . 3^2$, $X^n - R \equiv 0$ mod. 72 has

$$
\begin{array}{ll}
\text{for } n = 2, & (4 + 2)\,(2 + 3) = 30 \text{ roots,} \\
\text{“} \quad n = 3, & (1 + 2^2)\,(3 + 3) = 30 \quad \text{“} \\
\text{“} \quad n = 4, & (4 + 2^2)\,(2 + 3) = 40 \quad \text{“} \\
\text{“} \quad n = 5, & (1 + 2^2)\,(1 + 3) = 20 \quad \text{“} \\
\text{“} \quad n = 6, & (4 + 2^2)\,(6 + 3) = 72 \quad \text{“}
\end{array}
$$

or, in other words, all the 6^{th}-power residues of 72 are repetents. This number, 6, I shall call the period of the modulus 72. In general, the least value of n for which all the power residues become repetents of the modulus, k, I shall call the period of the modulus, k, and shall designate it by $P(k)$. The formula given for the number of the roots of $X^n - R \equiv 0$ mod. k shows us that $P(k)$ = the least common multiple of $\tau(a^t)$, $\tau(b^{t'})$, etc. For α, β, etc., then become $\tau(a^t)$, $\tau(b^{t'})$, etc., and the formula becomes

$$[\tau(a^t) + a^{t-1}]\,[\tau(b^{t'}) + b^{t'-1}] \ldots . [\tau(q^{t^{\text{IV}}}) + q^{t^{\text{IV}}-1}] = k.$$

The $P(k)^{\text{th}}$ powers of all the numbers prime to k are congruous to R_1, i. e. to unity, mod. k, as is shown in Serret's *Cours d'Algèbre Supérieure, tome* 2, *page* 51.

It will be noticed by examination of the different totients of k, given in § 1, that $\tau_s(k)$ is in every case either a divisor or a multiple of $P(k)$.

§ 6. *The Number of Roots of* $X^n - D \equiv 0$ *mod.* k.

D is supposed to be any n^{th}-power residue. Consider, first, the congruence $X^n - D \equiv 0$ mod. a^t. Divide the numbers represented by D into the following classes, viz.: (1) those prime to a^t; (2) those containing a^n and no higher power of a; (3) those that contain a^{2n} and no higher power of a; and so on, to lastly those that contain $a^{\theta n}$, where $\theta = E\left(\dfrac{t + n - 1}{n}\right)$, as in last section. Obviously all the n^{th}-power residues are included in this classification. We have, then, that the number of roots of $X^n - D \equiv 0$ mod. a^t is $\sum\limits_{r=0}^{r=\theta-1} \alpha_r a^{r(n-1)} + a^{t-\theta}$, where α_r is the g. c. d.* of n and $\tau(a^{t-rn})$, the different terms of the summation being the numbers of roots corresponding to the different classes of D's mentioned above.

First, $\alpha_r a^{r(n-1)}$ is the number of roots for any value of D which contains a^{nr} and no higher power of a, i. e. $D_{a^{nr}}$. For there are a^{t-r} numbers less than a^t which contain a^r, each of which may be represented by λa^r, where λ has any value less than a^{t-r}.

---

* Abbreviation for *greatest common divisor.*

Of course no number not included among the numbers, λa^r, can give rise to $D_{a^{nr}}$ as an n^{th}-power residue. Raising λa^r to the n^{th} power, $\lambda^n a^{nr}$, we see that its n^{th}-power residue mod. a^t is equal to the product of a^{nr} by the residue of λ^n mod. a^{t-nr}, since $a^t = a^{nr}a^{t-nr}$. That is, every n^{th} residue of mod. a^t may be decomposed in this way. If $D_{a^{nr}}$ be divided by a^{nr}, the quotient is prime to a by definition. Let H be this quotient. Then we know that $\lambda^n - H \equiv 0$ mod. a^{t-nr} has a_r roots, where $a_r =$ the g. c. d. of n and $\tau(a^{t-nr})$. But if we admit all the values of λ less than a^{t-r}, there are plainly $\dfrac{a^{t-r}}{a^{t-nr}} = a^{r(n-1)}$ times as many roots, and this is obviously the number of roots of $X^n - D_{a^{nr}} \equiv 0$ mod. a^t.

Secondly, the last term of the sum, $a^{t-\theta}$, is the number of roots for any value of D which contains $a^{n\theta}$, i. e. for $D = 0$. This is similar to the last term of the formula in the preceding section.

For $a^t = 2^t$, n even, and $t > 2$, $a_r =$ double the g. c. d. of n and $\frac{1}{2}\tau(2^{t-nr})$ [see preceding section].

For values of $n =$ or $> t$, $\theta = 1$, and the formula reduces to $a + a^{t-1}$, where a is the number of roots for any value of D prime to a^t, and $a^{t-\theta}$ is the number for $D = 0$, evidently the only two classes of D's which can occur as n^{th}-power residues.

The number of roots of $X^n - D \equiv 0$ mod. $a^t b^{t'} \ldots$ is

$$\left(\sum_{r=0}^{r=\theta-1} a_r a^{r(n-1)} + a^{t-\theta}\right)\left(\sum_{r'=0}^{r'=\theta'-1} \beta_{r'} b^{r'(n-1)} + b^{t'-\theta'}\right)\ldots$$

For simplicity let us consider a modulus of only two components, a^t, and $b^{t'}$. The proof would be the same for any number of components.

(1.) $a_r\beta_{r'}a^{r(n-1)}b^{r'(n-1)}$ is the number of roots for any value of D which contains $a^{nr}b^{nr'}$. There are $a^{t-r}b^{t-r'}$ numbers less than $a^t b^{t'}$ which contain $a^r b^{r'}$, each of which may be represented by $\lambda a^r b^{r'}$, where λ is any number less than $a^{t-r}b^{t'-r'}$. No number not included by $\lambda a^r b^{r'}$ can give rise to $D_{a^{nr}b^{nr'}}$ as an n^{th}-power residue. Raising $\lambda a^r b^{r'}$ to the n^{th} power, we see that its residue mod. $a^t b^{t'}$ is equal to the product of $a^{nr}b^{nr'}$ by the residue of λ^n mod. $a^{t-nr}b^{t'-nr'}$. Let $H = \dfrac{D_{a^{nr}b^{nr'}}}{a^{nr}b^{nr'}}$, prime to ab by definition. Then, since $\lambda^n - H \equiv 0$ mod. $a^{t-nr}b^{t'-nr'}$ has $a_r\beta_{r'}a^{r(n-1)}b^{r'(n-1)}$ roots, admitting all values of λ less than $a^{t-r}b^{t'-r'}$, it follows that the number of roots of $X^n - D_{a^{nr}b^{nr'}} \equiv 0$ mod. $a^t b^{t'}$ is the same.

(2.) $a^{t-\theta}\beta_{r'}b^{r'(n-1)}$ is the number of roots for any value of D which contains $a^{n\theta}b^{nr'}$. There are $a^{t-\theta}b^{t'-r'}$ numbers less than $a^t b^{t'}$ which contain $a^\theta b^{r'}$, and they may be represented by $\lambda a^\theta b^{r'}$ where λ has any value less than $a^{t-\theta}b^{t-r'}$. No number not included in $\lambda a^\theta b^{r'}$ can give rise to $D_{a^{n\theta}b^{nr'}}$ as an n^{th}-power residue.

Raising $\lambda a^\theta b^{r'}$ to the n^{th} power, we see that its residue mod. $a^t b^{t'}$ is equal to the product of $b^{nr'}$ by the residue of $\lambda^n a^{n\theta}$ mod. $b^{t'-nr'}$. Since $n\theta =$ or $> t$, every one of the numbers $\lambda a^\theta b^{r'}$ is a root of $X^n - D_{a^{n\theta}b^{nr'}} \equiv 0$ mod. a^t, and in order to determine how many roots there are for mod. $a^t b^{t'}$, it is necessary only to determine how many there are mod. $b^{t'}$. If $D_{a^{n\theta}b^{nr'}}$ be divided by $a^{n\theta}b^{nr}$, the quotient is prime to b by definition. Call it H. Then, since $\lambda^n - H \equiv 0$ mod. $b^{t'-nr'}$ has $a^{t-\theta}\beta_{r'}b^{r'(n-1)}$ roots for all values of λ less than $a^{t-\theta}b^{t'-r'}$, it follows that this is also the number of roots of $X^n - D_{a^{n\theta}b^{nr'}} \equiv 0$ mod. $a^t b^{t'}$.

(3.) That $b^{t'-\theta'}a_r a^{r(n-1)}$ is the number of roots of $X^n - D_{a^{nr}b^{n\theta}} \equiv 0$ mod. $a^t b^{t'}$ follows by symmetry from (2).

(4.) $a^{t-\theta}b^{t'-\theta'}$ is the number of roots of $X^n - D_{a^{n\theta}b^{n\theta'}} \equiv 0$ mod. $a^t b^{t'}$, as is obvious from what has gone before. Q. E. D.

For values of $n =$ or $>$ the largest exponent in $k = a^t b^{t'} c^{t''} \ldots$, we have $\theta = 1$, and the number of roots $= (\alpha + a^{t-1})(\beta + b^{t'-1})(\gamma + c^{t''-1}) \ldots$, the same as the formula in the preceding section, there being now 2^i classes of D's, each class including one of the repetents.

§ 7. *The Number of* n^th^*-ic Residues.*

The different terms of $\sum_{r=0}^{r=\theta-1} \tau(a^{t-r}) + a^{t-\theta}$ evidently express, respectively, (1) the number of numbers less than and prime to a^t; (2) the number which contain a and no higher power of a; (3) the number which contain a^2 and no higher power of a; and so on to, lastly, the number of those that contain a^θ.

It is also clear that for $k = a^t b^{t'} \ldots$ the different terms of

$$\left(\sum_{r=0}^{r=\theta-1} \tau(a^{t-r}) + a^{t-\theta}\right)\left(\sum_{r'=0}^{r'=\theta'-1} \tau(b^{t'-r'}) + b^{t'-\theta'}\right) \ldots$$

express, respectively (for simplicity let us consider only two components of the modulus), (1) the number of numbers less than $a^t b^{t'}$ which contain $a^r b^{r'}$ and no higher power of either a or b; (2) the number that contain $a^\theta b^{r'}$ and no higher power of b; (3) those that contain $a^r b^{\theta'}$ and no higher power of a; (4) the number of those containing $a^\theta b^{\theta'}$.

For the modulus a^t, then, $\tau(a^{t-r})$ is the number of numbers less than a^t which contain a^r and no higher power of a, each of which, therefore, has for its n^{th}-power residue some one of the numbers that we have represented by $D_{a^{nr}}$. The formula in § 6 shows us that any n^{th}-power residue belonging to a given class occurs the same number of times. Hence, if we divide the whole number belonging to a given class by the number of times each one occurs, we obtain the number of

different residues belonging to a given class. That is, $\frac{\tau(a^{t-r})}{a_r a^{r(n-1)}}$ is the number of different n^{th}-power residues, mod. a^t, belonging to the class $D_{a^{nr}}$.

Hence, dividing $\sum\limits_{r=0}^{r=\theta-1} \tau(a^{t-r}) + a^{t-\theta}$ by the expression in § 6 for the number of roots of $X^n - D \equiv 0$ mod. a^t, term by term, respectively, we obtain

THEOREM I. *The number of different* n^{th}*-power residues mod.* a^t *is* $\sum\limits_{r=0}^{r=\theta-1} \frac{\tau(a^t)}{a_r a^{rn}} + 1$, *where* $\theta = \mathrm{E}\left(\frac{t+n-1}{n}\right) = 1 + \mathrm{E}\left(\frac{t-1}{n}\right)$, *and* $a_r =$ *the g. c. d. of* n *and* $\tau(a^{t-rn})$.

In like manner, dividing the most general expression in the first part of this section by the most general expression in § 6 for the number of the roots of $X^n - D \equiv 0$ mod. $a^t b^{t'} \ldots$, we get

THEOREM II. The number of different n^{th}-power residues mod. $a^t b^{t'} \ldots$ is $\left(\sum\limits_{r=0}^{r=\theta-1} \frac{\tau(a^t)}{a_r a^{rn}} + 1\right)\left(\sum\limits_{r'=0}^{r'=\theta'-1} \frac{\tau(b^{t'})}{\beta_{r'} b^{r'n}} + 1\right) \ldots$

When $n =$ or $> t$, the formula becomes $\left(\frac{\tau(a^t)}{a} + 1\right)\left(\frac{\tau(b^{t'})}{\beta} + 1\right)\left(\frac{\tau(c^{t''})}{\gamma} + 1\right) \ldots$

When $n = P(k)$ the number of n^{th} residues becomes $(1+1)(1+1)(1+1) \ldots = 2^i$, as we have already seen.

Examples. Suppose $k = 3^5 . 5^4$ to find the whole number of different cubic residues.

$$\tau(a^t) = 3^4 . 2, \ \tau(b^{t'}) = 5^3 . 4, \ \theta = E\left(\frac{5+3-1}{3}\right) = 2, \ \theta' = E\left(\frac{4+3-1}{3}\right) = 2,$$

and we have the number of cubic residues $= \left(\frac{3^4 . 2}{3} + \frac{3^4 . 2}{3 . 3^3} + 1\right)\left(\frac{5^3 . 4}{1} + \frac{5^3 . 4}{1 . 5^3} + 1\right) = 28785.$

The number of 30^{th}-power residues $= \left(\frac{3^4 . 2}{6} + 1\right)\left(\frac{5^3 . 4}{10} + 1\right) = 1428.$

That the number of n^{th}-power residues mod. k is equal to the product of the numbers for the different components of k is also readily seen from the following.

If D' be an n^{th}-power residue mod. $k = a^t b^u \ldots q^z$, then $D' \equiv a'$ mod. a^t, $D' \equiv b'$ mod. $b^u, \ldots D' \equiv q'$ mod. q^z, where a' is some one of the n^{th}-power residues of mod. a^t, b' is some one of the n^{th}-power residues of mod. b^u, etc., etc. Then $D' \equiv a'R_{\bar{a}} + b'R_{\bar{b}} + \cdots + q'R_{\bar{q}}$. Hence the whole number of n^{th}-power residues is equal to the number of different ways of combining the different numbers represented by a' with those represented by $b', c', \ldots q'$, that is to say, is equal to the product of the numbers of n^{th}-power residues for the different components of k, a^t, b^u, etc. No two values of D thus obtained will be congruous to each other [§ 2, Theorem VIII.]. If $s = ab \ldots l$, the different s-totitive n^{th}-power resi-

dues mod. k will be obtained by combining, according to the formula, the different n^{th}-power residues mod. a^t which are a-totitives of a^t, with the different n^{th}-power residues mod. b^u which are b-totitives of $b^u, \ldots.$ with those mod. l^x which are l-totitives of l^x, with those mod. m^y which are prime to $m^y, \ldots.$ with those mod. q^z which are prime to q^z. That is, the number of n^{th}-power residues mod. k which are s-totitives of $k = A'B' \ldots. L'M \ldots. Q$, where $A', B', \ldots. Q$, represent respectively the different numbers described above. We have seen that the whole number of n^{th}-power residues mod. $a^t = \sum_{r=0}^{r=\theta-1} \frac{\tau(a^t)}{a_r a^{rn}} + 1$. That is, $A = \frac{\tau(a^t)}{a}$, and $A' = \sum_{r=1}^{r=\theta-1} \frac{\tau(a^t)}{a_r a^{rn}} + 1$, and so on for B, B', C, C', etc. Then the *whole* number of n^{th}-power residues mod. $k = a^t b^u \ldots. q^z$ is $(A + A')(B + B') \ldots. (Q + Q')$, where each term of the expansion denotes the number of residues belonging to that class of totitives of k whose subscript corresponds to the accented letters in the term. Thus the number of n^{th} residues which are ap-totitives of k is $A'BC \ldots. P'Q$.

§ 8. *The Junction of Fermat's and Wilson's Theorems.*

We know from the theory of indices [see Dirichlet's *Zahlentheorie*, § 30] that if $k = a^t, 2a^t$, or 4, where a is an odd prime number, the $\frac{\tau(k)}{\delta}$ δ^{th}-power residues of mod. k which are prime to k are congruous to $g^\delta, g^{2\delta}, \ldots. g^{\frac{\tau(k)}{\delta}\delta}$, where δ is a divisor of $\tau(k)$, and g is any primitive root of k. Hence, if x' be any one of the δ roots of $x^\delta \equiv g^\delta$ mod. k, and x'' any one of the δ roots of $x^\delta \equiv g^{2\delta}$ mod. k, etc., etc., we have the following: —

THEOREM I. $\left(x'x'' \ldots. x^{\left[\frac{\tau(k)}{\delta}\right]}\right)^\delta \equiv \left(-1\right)^{\frac{\tau(k)}{\delta}+1}$ *mod.* k, *where* $k = a^t, 2a^t$, *or* 4.
For, by multiplying together the congruences $x^\delta \equiv g^\delta$, $x^\delta \equiv g^{2\delta}$, etc., mod. k, we get

$$\left(x'x'' \ldots. x^{\left[\frac{\tau(k)}{\delta}\right]}\right)^\delta \equiv g^{\frac{\tau(k)}{2}\left(\frac{\tau(k)}{\delta}+1\right)} \equiv \left(-1\right)^{\frac{\tau(k)}{\delta}+1} \text{ mod. } k,$$

since $g^{\frac{\tau(k)}{2}} \equiv -1$ mod. k. Q. E. D. This is only another form of the well-known theorem that the product of the δ^{th} power residues is congruous to $(-1)^{\frac{\tau(k)}{\delta}+1}$ mod. k. [See Gauss, D. A., § 75]. This theorem becomes Fermat's when $\delta = \tau(k)$, for we then have $x^{\tau(k)} \equiv 1$ mod. k; and when $\delta = 1$ we have $(x'x'' \ldots. x^{[\tau(k)]}) \equiv -1$ mod. k, which is Wilson's theorem. It may be remarked that there are $\delta^{\frac{\tau(k)}{\delta}}$ different combinations of numbers less than k which satisfy the congruence of this theorem. Instead of -1 we may write $-R_1$.

For completeness I state what is obvious for $k = a^t$, viz. $\left(x'x'' \dots . x^{\left[\frac{\tau_a(k)}{\delta}\right]}\right)^\delta \equiv -R_a \equiv 0 \text{ mod. } k$, where $x', x'', \dots .$ are a-totitives of k, and this of course holds when $a = 2$. But if $a = 2$ x', x'', etc., be odd, we have

THEOREM II. If $k = 2^t$, $t > 2$, and δ be any divisor of $\tau(k)$, then

$$\left(x'x'' \dots . x^{\left[\frac{\tau(k)}{\delta}\right]}\right)^\delta \equiv 1 \text{ mod. } k.$$

For δ must be unity or some power of 2. When $\delta = 1$ the theorem becomes Wilson's theorem. When $\delta =$ a power of 2, we know that the $\frac{1}{2} \frac{\tau(k)}{\delta}$ δ^{th}-power residues which are prime to k are congruous to $g^\delta, g^{2\delta}, \dots . g^{\frac{1}{2}\frac{\tau(k)}{\delta}}$, where g is some one of those roots of $x^{\frac{1}{2}\tau(k)} \equiv 1$ mod. k which are not roots of a similar congruence of a less degree. Hence, if x' be any one of the 2δ roots of $x^\delta \equiv g_\delta$ mod. k [see Serret's *Cours d'Algèbre Supérieure*, § 324], x'' any other root of the same congruence, x''' any one of the 2δ roots of $x^\delta \equiv g^{2\delta}$ mod. k, x^{IV} any other root of the same congruence, etc., etc., we have by multiplication

$$\left(x'x'' \dots . x^{\left[\frac{\tau(k)}{\delta}\right]}\right)^\delta \equiv g^{2\delta} g^{4\delta} g^{6\delta} \dots . g^{\frac{\tau(k)}{\delta}\delta} \equiv g^{\frac{\tau(k)}{2}\left(\frac{\tau(k)}{2\delta}+1\right)} \equiv 1 \text{ mod. } k,$$

since $g^{\frac{\tau(k)}{2}} \equiv 1$ mod. k. Q. E. D. For $\delta = \tau(k)$ this becomes Fermat's theorem.

We can now prove the most general form of these theorems, bringing together the extended Fermat's and Wilson's theorems given in § 3.

THEOREM III. If $k = a^t b^u \dots . q^z$, and $\Delta =$ any divisor of $\tau\left(\frac{k}{\sigma}\right)$, then $\left(X_s' X_s'' \dots . X_s^{\left[\frac{\tau_s(k)}{\Delta}\right]}\right)^\Delta \equiv R_s$ mod. k; except when $\frac{k}{\sigma} = q^z, 2q^z$, or $= 4$, and $\frac{\sigma}{s}$ is at the same time an odd number, in which cases

$$\left(X_s' X_s'' \dots . X_s^{\left[\frac{\tau_s(k)}{\Delta}\right]}\right)^\Delta \equiv (-R_s)^{Q+1} \text{ mod. } k,$$

where Q is the number of the Δ^{th}-power residues of mod. q^z, mod. $2q^z$, or mod. 4, which are prime to the modulus. X_s', X_s'', etc., will be defined below.

If $s = ab \dots . l$, we have seen that the number of Δ^{th}-power residues mod. k which are s-totitives of k is $A'B' \dots . L'M \dots . Q$ [close of preceding section], where A' is the number of the Δ^{th}-power residues mod. a^t which contain a, $\dots . M$ is the number of those mod. m^y which are prime to m, etc., etc. Put $A'B' \dots . Q = \Omega$. When $\Delta = 1$, $\Omega = \tau_s(k)$, and when $\Delta =$ or $>$ the largest exponent in σ, $\Omega = M \dots . Q = \omega$, for brevity.

We have seen that, if D_s' be a Δ^{th}-power residue mod. k, $D_s' \equiv a'R_{\bar{a}} + b'R_{\bar{b}} + \dots . + q'R_{\bar{q}}$ mod. k, where a' is any Δ^{th}-power residue mod. a^t, etc., etc., for

$b', \ldots, q'$. Give a^{t-1} values to a', i. e. all its A' distinct values and $a^{t-1} - A'$ repetitions to make up the number to a^{t-1}. Give b^{u-1} values to $b', \ldots, l^{x-1}$ values to l' in the same way. Give to $m', \ldots, q'$, their $M, \ldots, Q$, distinct values, respectively. We shall thus get $\frac{\sigma}{s}\,\omega$ values of D'_s, including among them the Ω distinct values of D'_s. We know that there are ω incongruous values of D'_s mod. $\frac{k}{\sigma}$. Represent the product of these $\frac{\sigma}{s}\,\omega$ values of D'_s by ΠD_s. Then we have [see Theorem I. and § 2, Theorem VI.],

$$\Pi D_s \equiv \left(m'm'' \ldots m^{[M]}\right)^{\frac{\sigma}{s}N \ldots Q} R_{\bar{m}} + \ldots + \left(q'q'' \ldots q^{[Q]}\right)^{\frac{\sigma}{s}M \ldots P} R_{\bar{q}} \text{ mod. } k,$$

$$\therefore \Pi D_s \equiv \left(-1\right)^{(M+1)\frac{\sigma}{s}N \ldots Q} R_{\bar{m}} + \ldots + \left(-1\right)^{(Q+1)\frac{\sigma}{s}M \ldots P} R_{\bar{q}} \text{ mod. } k,$$

by Theorem I., whence if $\frac{\sigma}{s}$ be even, or if any two of the numbers $M, \ldots, Q$, be even, or if they be all odd, we have $\Pi D_s \equiv R_{\bar{m}} + \ldots + R_{\bar{q}} \equiv R_{\overline{m \ldots q}} = R_s$ [see § 2, Formula (C)]. In any case, we have

$$(\Pi D_s)^2 \equiv R_{\bar{m}} + \ldots + R_{\bar{q}} \equiv R_{\overline{m \ldots q}} = R_s \text{ mod } k.$$

If one, and only one, of the numbers $M, \ldots, Q$ be even, Δ is even, since $\tau(m^v)$, $\ldots, \tau(q^z)$, are even. Hence we have $\frac{\tau_s(k)}{\Delta} \div \frac{\sigma}{s}\,\omega = \frac{\mu \ldots \chi}{\Delta} = \epsilon$, for brevity, $=$ an even number. For $M = \frac{\tau(m^v)}{\mu}, \ldots, Q = \frac{\tau(q^z)}{\chi}$, where μ is the g. c. d. of Δ and $\tau(m^v)$, etc., etc., and if M, for instance, be the one that is even, μ will contain as high a power of 2 as Δ, while $\nu, \ldots, \chi$, will each contain at least the first power of 2, since the numbers which they divide are even.

Hence, if $D'_s, D''_s, \ldots, D^{[\Omega]}_s$, be the Ω distinct Δ^{th}-power residues mod. k which are s-totitives of k, and X'_s be any one of the roots of $X^\Delta_s \equiv D'_s$ mod. k, X''_s any one of the roots of $X^\Delta_s \equiv D''_s$ mod. k, etc., to $X^{[\Omega]}_s$ any one of the roots of $X^\Delta_s \equiv D^{[\Omega]}_s$ mod. k, and if $\frac{\sigma}{s}\,\omega - \Omega$ more roots be chosen in any way from these congruences, provided that $\frac{\sigma}{s} - \frac{\Omega}{\omega}$ be taken from each of the ω classes of congruences which are distinct with reference to the modulus $\frac{k}{\sigma}$, then, according to what has just been proved, we have by multiplication,

$\left(X'_sX''_s \ldots X^{\left[\frac{\sigma}{s}\omega\right]}_s\right)^\Delta \equiv \Pi D_s \equiv R_s$ mod. k, provided $\frac{\sigma}{s}$ is even, or at least two of the quantities $M, \ldots, Q$, are even, or if they are all odd. If these conditions be not fulfilled, then, since in this case ϵ is even, we may duplicate

the numbers X_s', X_s'', etc., taking each duplicate from the roots of the same congruence from which X_s', X_s'', etc., were taken respectively, and then have

$$\left(X_s'X_s'' \ldots . X_s^{\left[2\frac{\sigma}{s}\omega\right]}\right)^{\Delta} \equiv (\Pi D_s)^2 \equiv R_s \text{ mod. } k.$$

If, now, $\epsilon > 1$ in the first case and $\frac{\epsilon}{2} > 1$ in the second case, we have, by an ϵ-fold reduplication of the numbers, X_s', X_s'', etc., where the roots taken from any particular one of the congruences may be the same or different,

$$\left(X_s'X_s'' \ldots . X_s^{\left[\frac{\tau_s(k)}{\Delta}\right]}\right)^{\Delta} \equiv R_s \text{ mod. } k, \text{ since } R_s^n \equiv R_s.$$

This proves the theorem outside of the excepted cases, and these let us now consider.

For $\frac{k}{\sigma} = q^s$, we have, in the same way as before,

$$\Pi D_s \equiv \left(q'q'' \ldots . q^{[Q]}\right)^{\frac{\sigma}{s}} R_{\bar{q}} \equiv \left(-1\right)^{(Q+1)\frac{\sigma}{s}} R_{\bar{q}} \equiv R_{\bar{q}} \equiv R_s \text{ mod. } k,$$

where $\frac{\sigma}{s}$ is even; otherwise $\equiv (-R_s)^{Q+1}$ mod. k.

For $\frac{k}{\sigma} = 4$, we have $\Pi D_s \equiv \left(-R_{\bar{2}}\right)^{(Q+1)\frac{\sigma}{s}} \equiv \left(-R_s\right)^{(Q+1)\frac{\sigma}{s}}$, as before.

For $\frac{k}{\sigma} = 2\,q^s$, we get, in the same way as before,

$$\Pi D_s \equiv \left(h'h'' \ldots . h^{[H]}\right)^{\frac{\sigma}{s}Q} R_{\bar{2}} + \left(q'q'' \ldots . q^{[Q]}\right)^{\frac{\sigma}{s}H} R_{\bar{q}} \text{ mod. } k.$$

But since $\tau(2) = 1$, $H = 1$, and h', the Δ^{th}-power residue mod. 2 which is prime to 2, is also $= 1$. Hence, we have here $\Pi D_s \equiv R_{\bar{2}} + \left(-1\right)^{(Q+1)\frac{\sigma}{s}} R_{\bar{q}}$ mod. k. When $k =$ twice an odd number, as in this case, we have plainly $R_{\bar{2}} = \frac{1}{2}k$, for $\frac{1}{2}k$ contains all the components of k except 2. Hence $R_{\bar{2}} \equiv -R_{\bar{2}}$ mod. k. Hence, since $\frac{\sigma}{s}$ is odd, we have

$$\Pi D_s \equiv (-R_{\bar{2}} - R_{\bar{q}})^{Q+1} \equiv (-R_{\bar{2}\bar{q}})^{Q+1} \equiv (-R_s)^{Q+1} \text{ mod. } k.$$

In these excepted cases $\epsilon = \frac{\chi}{\Delta} = 1$, since Δ is any divisor of $\tau\left(\frac{k}{\sigma}\right)$, i. e. $\tau(q^s)$, or $\tau(2\,q^s)$, or $\tau(4)$, and χ is the g. c. d. of Δ and $\tau(q^s)$, or $\tau(4)$. Hence $\frac{\tau_s k}{\Delta} = \frac{\sigma}{s}\,\omega$, and we have proved $\left(X_s'X_s'' \ldots . X_s^{\left[\frac{\tau_s(k)}{\Delta}\right]}\right)^{\Delta} \equiv (-R_s)^{Q+1}$ mod. k, for the excepted

cases, the theorem being obvious when $\frac{k}{\sigma} = 1$, and for the non-excepted cases $\left(X_s' X_s'' \ldots . X_s^{\left[\frac{\tau_s(k)}{\Delta}\right]}\right)^{\Delta} \equiv R_s \bmod. k$. Q. E. D.

For $\Delta = 1$, we have the extended form of Wilson's theorem given in § 3.

For $\Delta = \tau\left(\frac{k}{\sigma}\right)$, we have $\left(X_s' X_s'' \ldots . X_s^{\left[\frac{\sigma}{s}\right]}\right)^{\tau\left(\frac{k}{\sigma}\right)} \equiv R_s$ or $(-R_s)^{Q+1}$, and since Ω in this case $= 1$, the numbers X_s', X_s'', etc., are all roots of $X_s^{\Delta} \equiv R_s$ mod. k. Hence we have $X_s^{\frac{\sigma}{s}\tau\left(\frac{k}{\sigma}\right)} \equiv R_s$, since $Q = 1$. Hence $X_s^{\tau_s(k)} \equiv R_s$ mod. k, the Fermatian theorem of § 3. Although the theorems of § 3 are included in the above theorem, it seemed best, for the sake of simplicity and clearness, to prove the special cases first.

SCHOLIUM. If $\Pi' D_s$ denote the product of all the Ω distinct n^{th}-power residues which are s-totitives of k, the above method of proof shows that $\Pi' D_s \equiv R_s$ mod. k if all the numbers $M, \ldots, Q$, be odd, or if at least two be even. In any case we have $(\Pi' D_s)^2 \equiv R_s$ mod. k.

It may be proved that, if $X^{\Delta} \equiv D$ mod. k, then $D^{\frac{P(k)}{\Delta}} \equiv R$ mod. k, where R denotes an undistinguished repetent. The proof is exactly the same as that of the ordinary theorem for prime moduli [Serret, *Alg. Sup.*, § 310], and need not be repeated here. The reciprocal of this does not hold, in general, for composite moduli. That is, it is not true, in general, that all the roots of $X^{\frac{P(k)}{\Delta}} - R \equiv 0$ mod. k are Δ^{th}-power residues, as may be easily seen. A brief consideration of the case where $\Delta = 2$ will close the present paper. We know that $X^{P(k)} - R \equiv 0$, or $(X^{\frac{P(k)}{2}} - R)(X^{\frac{P(k)}{2}} + R) \equiv 0$ mod. k, has k roots. All the quadratic residues of mod. k are roots of the first factor; the non-quadratic residues may be roots of either factor, or of neither, i. e. may be roots of the product of the two factors. For the particular case where the prime factors of k are each of the first degree, and of the form $2^c(2^r + 1) + 1$, where c is constant and r variable, application of the formulæ of § 5 and § 7 shows us that the number of roots of $X^{\frac{P(k)}{2}} - R \equiv 0$ mod. k is the same as the number of the quadratic residues mod. k, viz.: $[2^{c-1}(2^r + 1) + 1]\,[2^{c-1}(2^{r'} + 1) + 1] \ldots .$ So that in this case, just as in the case of prime moduli, all the roots of $X^{\frac{P(k)}{2}} - R \equiv 0$ mod. k are quadratic residues, and *vice versa*.

Linkages for x^m.

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By the following method of combining reciprocators (a Peaucellier cell with a modulus one) and bisectors (a pantograph with equal arms) a linkage for x^m may be obtained, m being any positive or negative integer or fraction.

Let the reciprocator be designated by R, the bisector by B, and the linkage for x^m by L^m. Then $L^{\frac{1}{m}}$ will designate the linkage for $x^{\frac{1}{m}}$, or the m^{th} root linkage, which may be obtained from that for x^m by transposing its arms.

If a and b be the distances of two points P_1 and P_2, measured in opposite directions, from a point O, or two points symmetrical with respect to O, and if the two poles of a B are made to coincide with P_1 and P_2, O' being the bisecting point, then it may be easily proved that $OO' = \frac{1}{2}(a - b)$.

Let $m = -n$, n being any positive integer or fraction. Then L^m may be obtained by combining L^n and one R, for $\frac{1}{x^n} = x^{-n} = x^m$.

Let $m = \frac{p}{q}$, p and q being positive integers. Then L^m may be obtained by combining L^p and $L^{\frac{1}{q}}$ for $(x^p)^{\frac{1}{q}} = x^{\frac{p}{q}} = x^m$.

Let $m = 2p - q$, p and q being integers and p positive. Then L^m may be obtained in two different ways by combining L^p, L^q, three R's and one B.

(a.) Add $-x^q$ and $+x^q$ to x^p, giving $x^p - x^q$ and $x^p + x^q$, and convert by two R's into $\frac{1}{x^p - x^q}$ and $\frac{1}{x^p + x^q}$. The B will then give

$$\frac{1}{2}\left[\frac{1}{x^p - x^q} - \frac{1}{x^p + x^q}\right] = \frac{x^q}{x^{2p} - x^{2q}}.$$

The remaining R gives $x^{2p-q} - x^q$, and adding x^q we have x^{2p-q} or x^m.

(b.) Subtract x^q from x^p, giving $x^p - x^q$. Convert by one R into $\frac{1}{x^p - x^q}$. By another R form $\frac{1}{x^p}$, and by the aid of the B subtract from $\frac{1}{x^p - x^q}$, giving $\frac{x^q}{2x^p(x^p - x^q)}$.

Fig. 1.

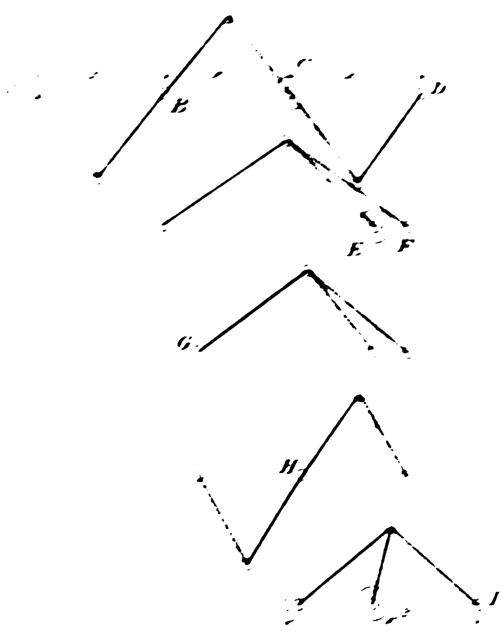


Fig. 2.

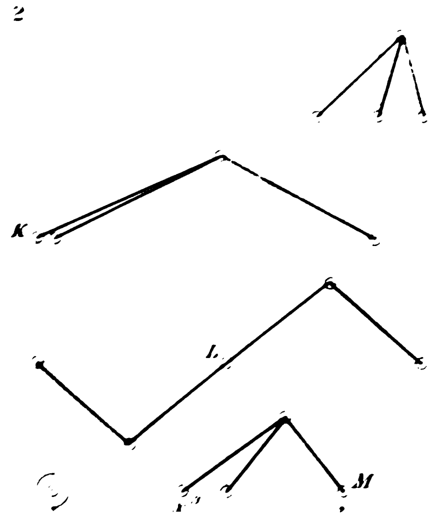


Fig. 3.

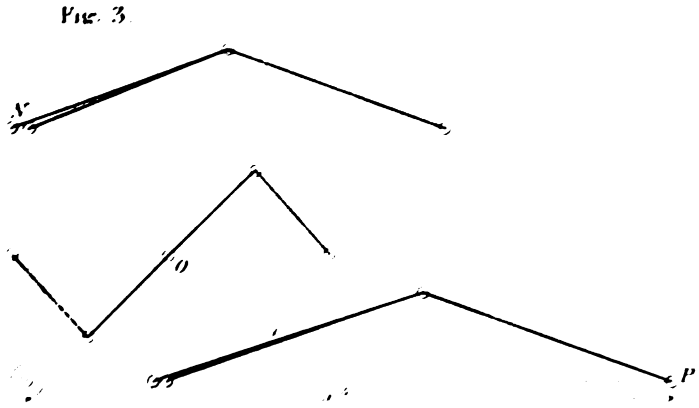


Fig. 4.

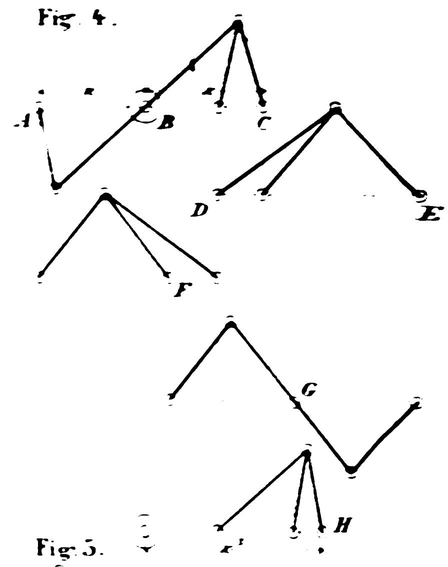


Fig. 5.

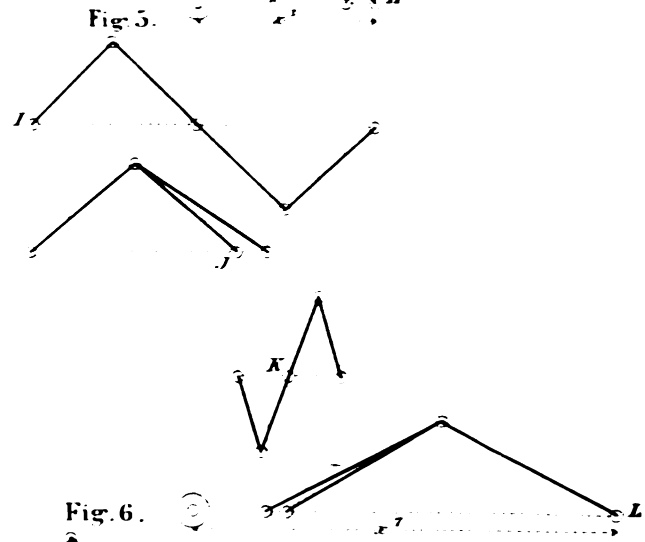
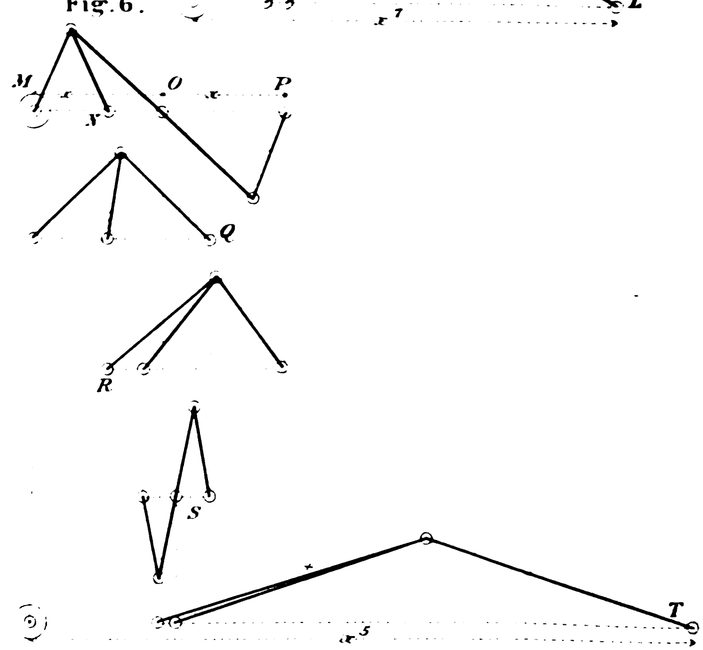


Fig. 6.



This may be converted into $x^{2p-q}-x^p$ by the last R, slightly modified, and adding x^p, we have x^{2p-q} or x^m.

The arrangement of the cells which renders it possible to add and subtract these various quantities will be best understood from the examples.

If $q=+1$, 0, or -1, $m=2n-1$, $2n$, or $2n+1$; and as m must be of one of these forms we can always obtain L^m from L^n by either of the above methods. When m is odd, L^m may be made to depend upon the linkage for an odd or an even power. By making $n=1$ we obtain $m=1$, 2, or 3. Making $n=2$ and 3 successively, we have $m=3, 4, 5$; 5, 6, 7, and so on, thus giving the linkages for every positive integral power.

In the figures the cells of the compound linkages have been separated for the sake of clearness, and only one half of each drawn. Points on the same vertical line should coincide. A small circle represents a joint, a large one a fixed point, and a heavy line a link. Each letter refers not only to the point in its immediate vicinity, but also to all points upon the same vertical line. A + or — sign indicates a positive or negative reciprocator.

Fig. 1 gives the linkage for x^2. Let $BC=CD=x$ and $CE=1$. The portion AB and similar extensions are only added when x^5 and higher powers are to be derived by the first method. Then $BE=x+1$ and $ED=x-1$. By the two R's $GE=\frac{1}{x-1}$ and $EF=\frac{1}{x+1}$. By the B,

$$HE=\frac{1}{2}\left[\frac{1}{x-1}-\frac{1}{x+1}\right]=\frac{1}{x^2-1}.$$

The last R gives $EI=x^2-1$ and $CI=EI+1=x^2$. It is not necessary that E should be a fixed point, it is sufficient to have E and C connected by a link of a length unity.

Fig. 2 gives the derivation of the linkage for x^3 from that for x^2 by adding three R's and one B. $DI=x^2-x$ and $IB=x^2+x$. The two R's give $DJ=\frac{1}{x(x-1)}$ and $KB=\frac{1}{x(x+1)}$. The B gives $CL=\frac{1}{2}\left[\frac{1}{x(x-1)}-\frac{1}{x(x+1)}\right]=\frac{1}{x(x^2-1)}$. The last R gives $CM=x^3-x$ and $BM=CM+x=x^3$. Let the two points C and E be connected by a link of a length unity, as before stated, and set free. Let B be made the fixed point, and we have a linkage in which the arm $BC=x$ and $BM=x^3$ measured in the same direction. It is necessary that B, C, and E should remain in the same straight line.

Fig. 3 gives the derivation of the linkage for x^5 from that for x^3 by adding two R's and one B. $AM=x^3+x$, and the first R gives $AN=\frac{1}{x^3+x}$,

$CL = \frac{1}{x^3 - x}$. The B gives $BO = \frac{1}{2}\left[\frac{1}{x(x^2-1)} - \frac{1}{x(x^2+1)}\right] = \frac{1}{x(x^4-1)}$. The last R gives $BP = x^5 - x$ and $AP = BP + x = x^5$. Setting free B and making A the fixed point, we have a linkage in which the arm $AB = x$ and $AP = x^5$ measured in the same direction.

Fig. 4 gives the derivation of the linkage for x^3 from those for x and $\frac{1}{x}$ by adding one B and three R's by the first method. $BC = x$, $BD = \frac{1}{x}$. $DC = \frac{x^2-1}{x}$ and the R gives $DE = \frac{x}{x^2-1}$. $DA = \frac{x^2+1}{x}$, and the R gives $DF = \frac{x}{x^2+1}$. The B gives $DG = \frac{1}{2}\left[\frac{x}{x^2-1} - \frac{x}{x^2+1}\right] = \frac{x}{x^4-1}$. The last R gives

$$DH = x^3 - \frac{1}{x} \text{ and } HB = HD + \frac{1}{x} = x^3.$$

If the second method had been used, we should have arrived at the well-known form for the cube devised by Professor Sylvester, consisting of only three cells, and the reason is this, $x^p = x$, $x^q = x^{-1}$ $\therefore \frac{1}{x^p} = x^q$; and as x^q is already at command, having been formed by the first R, we do not need the second cell in the transformation; also, after this change we do not need the B when it is necessary to form $\frac{x}{x^2-1} - \frac{1}{x}$, thus giving three cells instead of five. See Fig. 6 and its explanation.

Fig. 5 gives the derivation of the linkage for x^7 from those for x^3 and $\frac{1}{x}$ by adding a B to transfer a distance, two R's and one B. The third R being unnecessary, as $DG = \frac{1}{x^3 - \frac{1}{x}}$ had been formed in the linkage for x^3. $ID = x^3 + \frac{1}{x}$. The R gives $JD = \frac{x}{x^4+1}$. The B gives $DK = \frac{1}{2}\left[\frac{x}{x^4-1} - \frac{x}{x^4+1}\right] = \frac{x}{x^8-1}$. The last R gives $DL = x^7 - \frac{1}{x}$ and $BL = DL + \frac{1}{x} = x^7$.

Fig. 6 gives a form for x^5 consisting of five cells or thirty-two links. The first part of the transformation is conducted by the second method and the latter part by the first method. $MO = OP = x$. $MN = x - \frac{1}{x}$. The R gives $NQ = \frac{x}{x^2-1}$. $NO = \frac{1}{x}$. $OQ = \frac{x}{x^2-1} - \frac{1}{x} = \frac{1}{x(x^2-1)}$. (If this were inverted to a point Z to the right by a R, then ZM would equal x^3, thus giving the linkage for the cube devised by Professor Sylvester.) $NP = x + \frac{1}{x}$. The R gives $NR = \frac{x}{x^2+1}$ and $OR = \frac{1}{x} - \frac{x}{x^2+1} = \frac{1}{x(x^2+1)}$. The B then gives

$$OS = \frac{1}{2}\left[\frac{1}{x(x^2-1)} - \frac{1}{x(x^2+1)}\right] = \frac{1}{x(x^4-1)}.$$

The last R gives $OT = x^5 - x$ and $MT = OT + x = x^5$. Making M the fixed point, we have a linkage in which the arm $MO = x$ and $MT = x^5$ measured in the same direction.

It is unnecessary to give examples of the linkages for the higher powers, as their formation is a process as mechanical as writing out the successive terms of a series.

The above arrangements admit of a number of variations, some of which reduce the number of links.

DECEMBER, 1879.

The Strophoids.

By William Woolsey Johnson, *Annapolis, Md.*

The term *Strophoid* has been applied by French writers to a cubic curve, of which the symmetrical form has been discussed by Dr. James Booth under the name of the Logocyclic Curve. As this curve is one of the class considered in this paper, and as the term *Strophoid* is appropriate to the mode of generation of the whole class, I have ventured to use the word in a more extended signification, and define the strophoid as the *locus of the intersection of two straight lines which rotate uniformly about two fixed points in a plane.*

A and B being the fixed points, if we denote by θ and ϕ, the inclinations of the radii-vectores PA and PB, the direction AB being taken as that of the prime vector, we have by the definition

$$n\theta + m\phi = \alpha, \tag{1}$$

in which the ratio $m:n$, which determines the relative velocity of rotation, will in what follows be regarded as commensurable, so that m and n may be taken as integers prime to one another. Restricting m and n to positive values, (1) represents the case in which the lines rotate in opposite directions; and when they rotate in the same direction we may write

$$n\theta - m\phi = \alpha. \tag{2}$$

Taking A as the origin of rectangular co-ordinates, and AB as the axis of x,

$$x = \sqrt{(x^2 + y^2)} \cos\theta, \qquad y = \sqrt{(x^2 + y^2)} \sin\theta;$$

hence, if we put

$$(x + iy)^n = X_n + iY_n, \tag{3}$$

[so that $X_0 = 1$, $Y_0 = 0$; $X_1 = x$, $Y_1 = y$; $X_2 = x^2 - y^2$, $Y_2 = 2xy$, etc.], we have, by De Moivre's Theorem,

$$X_n = (x^2 + y^2)^{\frac{n}{2}} \cos n\theta, \qquad Y_n = (x^2 + y^2)^{\frac{n}{2}} \sin n\theta. \tag{4}$$

From this we have (r and s being positive),

$$\begin{aligned} X_{r+s} &= X_r X_s - Y_r Y_s, \\ Y_{r+s} &= Y_r X_s + X_r Y_s, \end{aligned} \tag{5}$$

and, if $r > s$,

$$\begin{aligned} (x^2+y^2)^s X_{r-s} &= X_r X_s + Y_r Y_s, \\ (x^2+y^2)^s Y_{r-s} &= Y_r X_s - X_r Y_s. \end{aligned} \tag{6}$$

If we denote by X'_m and Y'_m the results of putting $x - a$ in place of x in the values of X_m and Y_m, $\cos m\phi$ and $\sin m\phi$ are given by equations similar to (4); and, putting $q = \cot a$, we have from (1)

$$X_n X'_m - Y_n Y'_m - q\,(Y_n X'_m + X_n Y'_m) = 0, \tag{7}$$

when the lines rotate in opposite directions; and from (2)

$$X_n X'_m + Y_n Y'_m - q\,(Y_n X'_m - X_n Y'_m) = 0, \tag{8}$$

when the lines rotate in the same direction.

Differentiating (3) with respect to x, we find

$$\frac{d}{dx}(X_m + iY_m) = m\,(x+iy)^{m-1} = m\,(X_{m-1} + iY_{m-1}),$$

therefore

$$\frac{d}{dx} X_m = mX_{m-1}, \qquad \frac{d}{dx} Y_m = mY_{m-1};$$

hence, developing X'_m and Y'_m,

$$X'_m = X_m - maX_{m-1} + \frac{m\,(m-1)}{1.2}\,a^2 X_{m-2} - \ldots\ldots + (-1)^m a^m,$$

$$Y'_m = Y_m - maY_{m-1} + \ldots\ldots - (-1)^m ma^{m-1} y.$$

Substituting these values and making use of (5), (7) becomes

$$X_{n+m} - qY_{n+m} - ma[X_{n+m-1} - qY_{n+m-1}] + \ldots + (-1)^m a^m [X_n - qY_n] = 0, \tag{9}$$

the rectangular equation when the lines rotate in opposite directions.

Similarly substituting in (8) and making use of (6), we have, if $n > m$,

$$\begin{aligned} (x^2+y^2)^m\,[X_{n-m} - qY_{n-m}] - ma\,(x^2+y^2)^{m-1}\,[X_{n-m+1} - qY_{n-m+1}] + \ldots\ldots \\ \ldots\ldots + (-1)^m a^m\,[X_n - qY_n] = 0; \end{aligned} \tag{10}$$

but if $n < m$, the first terms of this equation take a different form, and (remembering that $X_0 + qY_0 = 1$), we have

$$(x^2+y^2)^n\left\{X_{m-n}+qY_{m-n}-ma\left[X_{m-n-1}+qY_{m-n-1}\right]+\ldots\ldots+(-1)^{m-n}\frac{m(m-1)\ldots\ldots(n+1)}{(m-n)!}a^{m-n}\right\}$$
$$+(-1)^{m-n+1}\frac{m(m-1)\ldots\ldots n}{(m-n+1)!}a^{m-n+1}(x^2+y^2)^{n-1}\left[X_1-qY_1\right]\ldots\ldots+(-1)^m a^m\left[X_n-qY_n\right]=0. \quad (11)$$

(10) and (11) are therefore forms of the rectangular equation when the lines rotate in the same direction.

The expressions $X_n - qY_n$, etc., which occur in these equations, are the products of linear factors which are all real; for, putting

$$X_n - qY_n = 0,$$

we have from (4)

$$\tan n\theta = \tan a,$$

whence

$$\theta = \frac{a}{n},\ \frac{a+\pi}{n},\ \ldots\ldots\ \frac{a+(n-1)\pi}{n},$$

values which determine n different real factors. Thus (9) indicates $n+m$ real asymptotes; and (10) and (11) indicate $n-m$ or $m-n$ real asymptotes, the remaining points at infinity being the circular points. Each of the equations also indicates an n-tuple point at A, at which n real tangents make equal angles with one another.

But the tangents at A and B and the asymptotes are readily determined geometrically as follows: When ϕ is a multiple of π, the line BP coincides with AB and

$$\theta = \frac{a+k\pi}{n}; \qquad (12)$$

unless θ is now a multiple of π (which can only happen when a is a multiple of π) the point P is at A, and the value of θ gives the inclination of a tangent at A. In like manner there are m tangents at B, whose inclinations are

$$\phi = \frac{a+k\pi}{m}. \qquad (13)$$

When $\theta-\phi$ is a multiple of π, the rotating lines are parallel and P is at infinity; hence

$$\theta = \frac{a+k\pi}{n+m} \qquad (14)$$

gives the inclinations of the asymptotes. If we put $AP=r$, $BP=r'$, the distances of the point at infinity from the parallel lines are plainly $rd\theta$ and $r'd\phi$; but from (1)

$$\frac{d\theta}{d\phi} = -\frac{m}{n}, \qquad (15)$$

and since at infinity $r = r'$, these distances have the ratio $m : n$. Therefore every asymptote passes through a point C on AB which divides AB (internally if m and n are positive) so that

$$\frac{AC}{CB} = \frac{m}{n}.$$

If, at a finite point of the curve, we denote the angles between the tangent and the radii-vectores r, r' by ψ, ψ', we have

$$\sin\psi = \frac{rd\theta}{ds}, \qquad \sin\psi' = \frac{r'd\phi}{ds},$$

whence, by (15),

$$\frac{\sin\psi}{\sin\psi'} = -\frac{mr}{nr'}.$$

Therefore, to construct the tangent at a given point P, lay off on PA and PB distances PQ, PR proportional to nPB and mPA respectively; bisect QR in T, then PT is the tangent. These results are, of course, applicable to equation (2) in which m is negative.

When $\alpha = 90^\circ$, the expressions $X_n - qY_n$, etc., in (9), (10), and (11) reduce to X_n, etc., and since X_n is an even function of y, we have a curve symmetrical to AB, which may be called a *right strophoid.*

When $\alpha = 0$, the expressions $X_n - qY_n$, etc., must be replaced by Y_n, etc., and since y is a factor of Y_n, the curve breaks up into the line $y = 0$, and a curve of the $(n + m - 1)^{\text{th}}$ degree, which may be called a *substrophoid.*

The substrophoid is, of course, symmetrical to the axis; the number of tangents at A and at B, as well as of asymptotes, is reduced by unity, and these tangents and asymptotes make equal angles with one another and the axis (which is neither a tangent nor an asymptote). The curve cuts the axis in a point D corresponding to $\theta = 0$, $\phi = 0$. At this point an element of arc subtends angles at A and B proportional to the rates of rotation; that is, D divides AB (internally when m and n have the same sign) so that

$$\frac{AD}{DB} = \frac{n}{m}.$$

When the lines rotate in opposite directions, the curve consists of infinite branches without loops; for it is evident that in passing from one position in which the lines are parallel to the next, one and only one of the lines passes through coincidence with the axis. As special cases, we have from (9), when $n = 1$ and $m = 1$, the rectangular hyperbola

$$x^2 - y^2 - 2qxy - a(x - qy) = 0,$$

A and B being extremities of a diameter. The substrophoid in this case is the straight line

$$2x - a = 0.$$

When $n = 2$, $m = 1$, we have

$$x^3 - 3xy^2 - q(3x^2y - y^3) - a(x^2 - y^2 - 2qxy) = 0;$$

the substrophoid being the hyperbola

$$3x^2 - y^2 - 2ax = 0,$$

of which $AD = \frac{2}{3}a$ is the tranverse axis.

When the lines rotate in the same direction, supposing $n > m$, so that the more rapid rotation takes place at B, it is evident that in passing in either direction from a position in which the lines are parallel, BP will come into coincidence with the axis before AP does, that is, P will arrive at A before it arrives at B. Hence the curve consists of m loops between A and B with infinite branches extending from A to the $m - n$ asymptotes. As special cases, we have from (10), when $n = 1$ and $m = 1$, the circle

$$x^2 + y^2 - a(x - qy) = 0$$

passing through A and B.

When $n = 2$ and $m = 1$, (10) gives

$$(x^2 + y^2)(x - qy) - a(x^2 - y^2 - 2qxy) = 0,$$

which is the cubic alluded to in the first paragraph. The right strophoid in this case is Dr. Booth's Logocyclic Curve

$$(x^2 + y^2)x - a(x^2 - y^2) = 0,$$

and the substrophoid is the circle

$$x^2 + y^2 - 2ax = 0,$$

with centre at B. If $n = 1$ and $m = 2$, we obtain from (11) another equation of the same curve; viz.,

$$(x^2 + y^2)(x + qy - 2a) + a^2(x - qy) = 0:$$

the right strophoid being

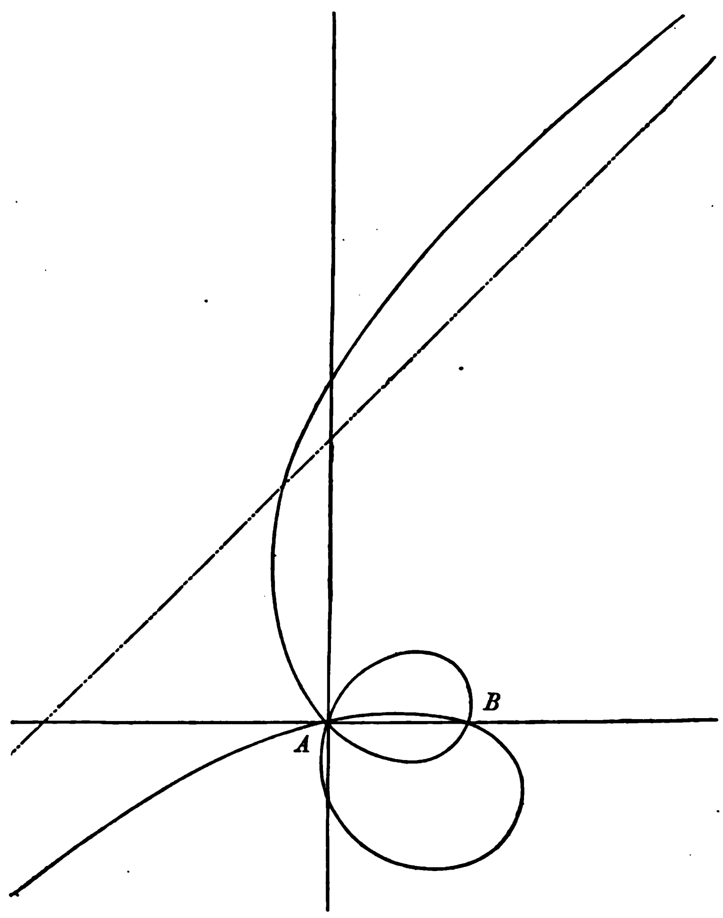
$$(x^2 + y^2)(x - 2a) + a^2x = 0,$$

and the substrophoid the circle

$$x^2 + y^2 - a^2 = 0.$$

The accompanying diagram is constructed for the case in which $n = 3$, $m = 2$ and $\alpha = 45°$, or $3\,\theta - 2\,\phi = \frac{1}{4}\pi$; its rectangular equation, therefore, is, from (10),

$$(x^2 + y^2)^2 (x - y) - 2\,a\,(x^2 + y^2)(x^2 - y^2 - 2\,xy) + a^2 (x^3 - 3\,xy^2 - 3\,x^2y + y^3) = 0.$$



The corresponding substrophoid is a case of the limaçon which is sometimes called the "trisectrix." The mode of employing this curve to trisect an angle is indicated by the equation $3\,\theta = 2\,\phi$.

On the Ratio between Sector and Triangle in the Orbit of a Celestial Body.

BY ORMOND STONE, *Cincinnati, Ohio.*

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1. THIS ratio may be expressed by the formula

$$\frac{1}{\eta} = \frac{rr' \sin(v' - v)}{\sqrt{p\tau}} = \frac{\sin 2\theta}{2\theta}, \tag{1}$$

where the mass of the body is neglected, r and r' are the radii vectores, v and v' the corresponding true anomalies, p the semi-parameter, and τ the product of the intervening time and the constant of the solar system.

By Taylor's theorem, including terms of the fourth order,

$$v = v_0 - \frac{1}{2}\frac{dv_0}{d\tau}\tau + \frac{1}{8}\frac{d^2v_0}{d\tau^2}\tau^2 - \frac{1}{48}\frac{d^3v_0}{d\tau^3}\tau^3 + \frac{1}{384}\frac{d^4v_0}{d\tau^4}\tau^4 - \ldots,$$

$$v' = v_0 + \frac{1}{2}\frac{dv_0}{d\tau}\tau + \frac{1}{8}\frac{d^2v_0}{d\tau^2}\tau^2 + \frac{1}{48}\frac{d^3v_0}{d\tau^3}\tau^3 + \frac{1}{384}\frac{d^4v_0}{d\tau^4}\tau^4 + \ldots,$$

where v_0 is the true anomaly corresponding to the mean of the times;

$$\therefore\ v' - v = \frac{dv_0}{d\tau}\tau + \frac{1}{24}\frac{d^3v_0}{d\tau^3}\tau^3 + \ldots.$$

$$\therefore\ \sin(v' - v) = \frac{dv_0}{d\tau}\tau + \left(\frac{1}{24}\frac{d^3v_0}{d\tau^3} - \frac{1}{6}\left(\frac{dv_0}{d\tau}\right)^3\right)\tau^3.$$

The well-known expressions $\frac{dv_0}{d\tau} = \frac{\sqrt{p}}{r_0^2}$ and $\frac{p}{r_0} = 1 + e\cos v_0$ give, by successive differentiation,

$$\frac{d^2r_0}{d\tau^2} = \frac{p - r_0}{r_0^3},$$

$$\frac{d^3v_0}{d\tau^3} = \frac{\sqrt{p}}{r_0^2}\left(\frac{6}{r_0^2}\left(\frac{dr_0}{d\tau}\right)^2 - \frac{2}{r_0}\frac{d^2r_0}{d\tau^2}\right),$$

whence by substitution, including terms of the third order,

$$\frac{\sin(v'-v)}{\sqrt{pr}}=\frac{1}{r_0^3}\left[1+\left(\frac{1}{4r_0^2}\left(\frac{dr_0}{dr}\right)^2-\frac{3p-r_0}{12r_0^4}\right)\tau^2+\ldots.\right]. \qquad (2)$$

Developing r and r' in the same manner as v and v' were developed, and multiplying the results, we have

$$rr'=r_0^2\left[1-\left(\frac{1}{4r_0^2}\left(\frac{dr_0}{dr}\right)^2-\frac{p-r_0}{4r_0^4}\right)\tau^2+\ldots.\right]. \qquad (3)$$

The product of (2) and (3) gives

$$\frac{1}{\eta}=1-\frac{\tau^2}{6r_0^3}+\ldots.=1-\frac{4\theta^2}{6}+\ldots.;$$

$$\therefore 2\theta=\frac{\tau}{r_0^{\frac{3}{2}}}+\ldots.=\frac{\tau}{(rr')^{\frac{3}{4}}}+\ldots.=\frac{2^{\frac{3}{2}}\tau}{(r+r')^{\frac{3}{2}}}+\ldots..$$

2. For a closer approximation we have the well-known equations

$$\frac{\eta^2}{m}=\frac{1}{l+\sin^2\frac{1}{2}g},$$

$$\frac{\eta^2}{m}(\eta-1)=\frac{2g-\sin 2g}{\sin^3 g},$$

in which

$$l=\frac{\sin^2\frac{1}{2}\gamma}{\cos\gamma},\qquad m=\frac{\theta_0^2}{2\cos^3\gamma},\qquad \cos\gamma=\frac{2\sqrt{rr'}}{r+r'}\cos\tfrac{1}{2}(v'-v),\qquad \theta_0^2=\frac{2\tau^2}{(r+r')^3}$$

and g is the half difference of the eccentric anomalies.

For the first of these equations we may write

$$x=\sin^2\tfrac{1}{2}g=\frac{m}{\eta^2}-l,$$

or by substitution,

$$x=\frac{1}{8}\frac{\theta_0^2}{\theta^2}\frac{\sin^2 2\theta}{\cos^3\gamma}-\frac{\sin^2\frac{1}{2}\gamma}{\cos\gamma}.$$

In the same manner the second becomes

$$\frac{\theta^2}{\theta_0^2}\cdot\frac{\cos^3\gamma}{\cos^3\theta}\cdot\frac{2\theta-\sin 2\theta}{\sin^3\theta}=\frac{2g-\sin 2g}{\sin^3 g}.$$

Gauss has developed the second member of the latter equation in a series arranged according to the ascending powers of x, as follows:

$$\frac{2g-\sin 2g}{\sin^3 g}=\frac{4}{3}\left(1+\frac{6}{5}x+\frac{6}{5}\cdot\frac{8}{7}x^2+\ldots.\right),$$

whence, if we develop $\frac{2\theta - \sin 2\theta}{\sin^3 \theta}$ in a similar series arranged according to the ascending powers of $z = \sin^2 \frac{1}{2}\theta$, we shall have

$$\theta^2 \cos^3 \gamma \left(1 + \tfrac{6}{5} z + \ldots.\right) = \theta_0^2 \cos^3 \theta \left(1 + \tfrac{6}{5} x + \ldots.\right),$$

or, dividing by $\left(1 + \frac{6}{5} z + \ldots.\right)\left(1 + \frac{6}{5} x + \ldots.\right)$,

$$\theta^2 \cos^3 \gamma \left(1 - \tfrac{6}{5} x + \ldots.\right) = \theta_0^2 \cos^3 \theta \left(1 - \tfrac{6}{5} z + \ldots.\right).$$

Substituting the values of x and z, and reducing,

$$\theta^2 \cos^2 \gamma \left(1 - \tfrac{4}{5} \sin^2 \tfrac{1}{2}\gamma + \ldots.\right) = \theta_0^2 \cos^2 \theta \left(1 - \tfrac{4}{5} \sin^2 \tfrac{1}{2}\theta + \ldots.\right),$$

whence, approximately,

$$\theta = \theta_0 \left(\frac{\cos \theta}{\cos \gamma}\right)^{12}.$$

This formula includes terms of the same order as those included in Hansen's method, and, if employed in connection with a table giving the logarithms of the ratios between sines and arcs, is rather more convenient.

Centre of Gravity of Surface and Solid of Revolution.

By E. W. HYDE, *University of Cincinnati.*

THE formulæ derived in this paper are for the most general case of the revolution of any curve, plane or tortuous, about any axis, through any angle.

Let $\rho = \phi(t) = \phi$ (t being omitted for brevity) be the equation of any curve, and let ϵ be a unit vector in any direction; then

$$\rho = \epsilon^{\frac{\theta}{\pi}} \phi \epsilon^{-\frac{\theta}{\pi}} \tag{1}$$

will be the equation of a surface of revolution formed by revolving $\phi(t)$ about a line through the origin in the direction ϵ.

As the origin may be moved to any point by introducing a constant vector into $\phi(t)$, and as any direction may be chosen for ϵ, this equation is perfectly general.

1st. The *surface* element is $TVD_\theta\rho D_t\rho \,.\, d\theta dt$.

$$D_\theta\rho = \epsilon^{\frac{2\theta}{\pi}} V\epsilon\phi, \quad \text{and} \quad D_t\rho = \epsilon^{\frac{\theta}{\pi}} \phi' \epsilon^{-\frac{\theta}{\pi}};$$

hence

$$TVD_\theta\rho D_t\rho = TV\epsilon^{\frac{2\theta}{\pi}} V\epsilon\phi\epsilon^{\frac{\theta}{\pi}} \phi' \epsilon^{-\frac{\theta}{\pi}} = \sqrt{\phi'^2 V^2\epsilon\phi - S^2\epsilon\phi\phi'} = TV\phi' V\epsilon\phi,$$

as may be shown by expanding and reducing. Therefore the surface is

$$S = \int_{t_1}^{t_2}\int_{\theta_1}^{\theta_2} d\theta dt TV\phi' V\epsilon\phi = (\theta_2 - \theta_1)\int_{t_1}^{t_2} dt TV\phi' V\epsilon\phi. \tag{2}$$

We have then for the centre of gravity, if the density be uniform,

$$\bar{\rho} = \int\int \epsilon^{\frac{\theta}{\pi}} \phi \epsilon^{-\frac{\theta}{\pi}} TV\phi' V\epsilon\phi \,.\, d\theta dt \div \int\int TV\phi' V\epsilon\phi \,.\, d\theta dt; \tag{3}$$

or, integrating for θ from θ_1 to θ_2,

$$\bar{\rho} = \int [\theta\epsilon^{-1} S\epsilon\phi - \epsilon^{\frac{2\theta}{\pi}} V\epsilon\phi]_{\theta_1}^{\theta_2} TV\phi' V\epsilon\phi \,.\, dt \div (\theta_2 - \theta_1) \int TV\phi' V\epsilon\phi \,.\, dt. \tag{4}$$

If the integration be from 0 to 2π, equation (4) becomes

$$\rho = \epsilon^{-1} \int S\epsilon\phi TV\phi' V\epsilon\phi . dt \div \int TV\phi' V\epsilon\phi . dt. \tag{5}$$

If the generating curve lie in a plane through the origin and the vector ϵ about which it is revolved, we have $S\epsilon\phi\phi' = 0$, and therefore $TV\phi' V\epsilon\phi = T\phi' V\epsilon\phi$, so that in that case this latter expression may be substituted for the former.

2d. *Volume.* Writing $\sigma = u\rho = u\epsilon^{\frac{\theta}{\pi}}\phi\epsilon^{-\frac{\theta}{\pi}}$, we have

$$V = -\int\int\int SD_u\sigma D_\theta\sigma D_t\sigma . dud\theta dt = -\int\int\int u^2 S\epsilon^{\frac{\theta}{\pi}}\phi\epsilon^{-\frac{\theta}{\pi}} VD_\theta\rho D_t\rho . dud\theta dt,$$

which on expansion and reduction becomes,

$$V = \int\int\int u^2 S . \phi\phi' V\epsilon\phi . dud\theta dt; \tag{6}$$

and for the centre of gravity

$$\bar{\rho} = \int\int\int u^3 \epsilon^{\frac{\theta}{\pi}}\phi\epsilon^{-\frac{\theta}{\pi}} S . \phi\phi' V\epsilon\phi . dud\theta dt \div \int\int\int u^2 S . \phi\phi' V\epsilon\phi . dud\theta dt. \tag{7}$$

If we integrate for u from 0 to 1, and for θ from θ_1 to θ_2, we have

$$\rho = \tfrac{1}{4}\int [\theta\epsilon^{-1} S\epsilon\phi - \epsilon^{\frac{2\theta}{\pi}} V\epsilon\phi]_{\theta_1}^{\theta_2} S . \phi\phi' V\epsilon\phi . dt \div \tfrac{1}{3}(\theta_2 - \theta_1)\int S . \phi\phi' V\epsilon\phi . dt. \tag{8}$$

If $\theta_1 = 0$ and $\theta_2 = 2\pi$, equation (10) becomes

$$\bar{\rho} = \tfrac{1}{4}\epsilon^{-1}\int S\epsilon\phi S . \phi\phi' V\epsilon\phi . dt \div \tfrac{1}{3}\int S . \phi\phi' V\epsilon\phi . dt. \tag{9}$$

Formulæ (7), (8), and (9) are unchanged when the generating curve lies in a plane through the axis, though in that case $TV\phi\phi' V\epsilon\phi$ may be substituted for $S . \phi\phi' V\epsilon\phi$.

As an example of these formulæ let us take the hyperboloid of one sheet generated by the revolution of a right line about an axis which it does not intersect.

Let $\phi(t) = a + t\beta$; then the hyperboloid will be

$$\rho = \epsilon^{\frac{\theta}{\pi}}(a + \beta t)\epsilon^{-\frac{\theta}{\pi}}. \tag{10}$$

Let $Sa\epsilon = Sa\beta = 0$, and $T\beta = 1$; then

$$TV\phi' V\epsilon\phi = \sqrt{\phi'^2 V^2\epsilon\phi - S^2\epsilon\phi\phi'} = \sqrt{-a^2 - S^2\epsilon a\beta - t^2 V^2\epsilon\beta} = TV\epsilon\beta\sqrt{a^2 + t^2},$$

if we write
$$\frac{T^2 a - S^2\epsilon a\beta}{T^2 V\epsilon\beta} = a^2; \tag{11}$$

$$\therefore \rho = \int [\theta t\epsilon^{-1} S\epsilon\beta - \epsilon^{\frac{2\theta}{\pi}} V\epsilon(a + \beta t)]_{\theta_1}^{\theta_2} dt\sqrt{a^2 + t^2} \div (\theta_2 - \theta_1)\int dt\sqrt{a^2 + t^2}$$

$$= \left[\epsilon^{-1} S\epsilon\beta - \frac{(\epsilon^{\frac{2\theta_2}{\pi}} - \epsilon^{\frac{2\theta_1}{\pi}}) V\epsilon\beta}{\theta_2 - \theta_1}\right] \frac{\int t dt\,\sqrt{a^2+t^2}}{\int dt\,\sqrt{a^2+t^2}} - \frac{(\epsilon^{\frac{2\theta_2}{\pi}} - \epsilon^{\frac{2\theta_1}{\pi}}) V\epsilon a}{\theta_2 - \theta_1}. \qquad (12)$$

The integration of this equation will give the centre of gravity of a strip of the surface of the hyperboloid lying between two positions of the generatrix at an angular distance apart of $\theta_2 - \theta_1$.

For the solid we have

$$S\phi\phi' V\epsilon\phi = S(a + \beta t)\,\beta V\epsilon(a + \beta t) = a^2 S\epsilon\beta;$$

so that

$$\bar{\rho} = \tfrac{1}{4}\int_0^{t_1} [\theta t \epsilon^{-1} S\epsilon\beta - \epsilon^{\frac{2\theta}{\pi}} V\epsilon(a + \beta t)]_{\theta_1}^{\theta_2} dt \div \tfrac{1}{8}(\theta_2 - \theta_1)\, t_1,$$

$$= \frac{3}{4}\left[\frac{1}{2} t_1 \epsilon^{-1} S\epsilon\beta - \frac{1}{\theta_2 - \theta_1}(\epsilon^{\frac{2\theta_2}{\pi}} - \epsilon^{\frac{2\theta_1}{\pi}})\, V\epsilon\left(a + \frac{1}{2} t_1 \beta\right)\right]. \qquad (13)$$

This gives the centre of gravity of the solid bounded by the plane $S\epsilon\rho = 0$, the hyperboloid, a cone with its vertex at the origin cutting the hyperboloid in the circle $\rho = \epsilon^{\frac{\theta}{\pi}} \phi(t_1)\, \epsilon^{-\frac{\theta}{\pi}}$, and two positions of the plane through the origin and the generating line given by θ_1 and θ_2.

It will be noticed on comparison that the formulæ derived in this article do not agree with equations (23), (25), and (26) of Mr. Stringham's article in No. 3, Vol. II., of this Journal. Those equations appear to be incorrect for the reason that, though the integration has already been performed with regard to one or more variables, ρ is still taken as the arm of the element, which *should* be the vector from the origin to the centre of gravity of the element. It would follow from each of these equations, since they are independent of ϕ, that the barycentric vector is independent of the angular distance through which the generating curve has been revolved, which is certainly not the fact.

On a Point in the Theory of Vulgar Fractions.

By J. J. Sylvester.

The reciprocal of an integer I call a simple fraction; any other fraction, whether rational or irrational, may be termed complex; but it is to be understood that only proper fractional quantities of either sort, i. e. fractions greater than zero and less than unity, will be considered in what follows.

Suppose Q to represent any fractional quantity; if Q lies between $\frac{1}{u_0 - 1}$ and $\frac{1}{u_0}$, we may make $Q = \frac{1}{u_0 + \delta} + Q'$, where δ is zero or a positive integer, and Q' will continue a proper fraction, which in like manner may be resolved into $\frac{1}{u_1 + \delta_1} + Q''$, and so on continually.

But if we make $\delta_0, \delta_1, \ldots$ each zero, the process of expansion becomes determinate. Any such determinate representation of a fractional quantity I shall term a *sorites*. It is obvious that in expanding a given fraction under the form of a sorites, the successive denominators, which I shall call the ***elements***, may be obtained by a process of division; if the fraction to be expanded is rational, the real divisor will be an integer which continually decreases,* and consequently every complex rational fraction can be expanded (and only in one way) under the form of a finite sorites.

The elements of a sorites are analogous to the partial quotients of a regular continued fraction; but there is this difference between the two cases, that whilst the latter quantities are perfectly arbitrary, the elements in question are subject to a certain law which I shall proceed to examine.

Let $n, p, q, \ldots r, s, \ldots t, u$ be the elements of a sorites. It is clear that the last remainder being the reciprocal of $\frac{1}{t} + \frac{1}{u}$, we must have $\frac{1}{t} + \frac{1}{u} < \frac{1}{t-1}$, that is to say, u greater than $t^2 - t$, i. e. u is equal to or greater than $t^2 - t + 1$. Again, if we look to the residue which gives birth to the element r, that must be of the form $\frac{1}{s - \epsilon}$, where ϵ is some fraction, and we must now have

* See examples of development of sorites, page 335.

$\frac{1}{r} + \frac{1}{s-\epsilon} < \frac{1}{r-1}$, or $s - \epsilon$ equal to or greater than $r^2 - r$. Hence s is equal to or greater than $r^2 - r + 1$, so that the relation between any two contiguous elements is the same, whether they are or are not the final two; and if u_x, u_{x+1} be any two consecutive integers in a series, the one necessary and sufficient condition for the possibility of the existence of the sorites, of which those terms shall be elements, is that we must have for all values of x, u_{x+1} equal to or greater than $u_x^2 - u_x + 1$.

If u_{x+1} is throughout equal to $u_x^2 - u_x + 1$, we obtain a series which may be termed a limiting sorites.

It is obvious that any simple fraction $\frac{1}{u_0 - 1}$ may be expanded under the form of an infinite sorites, of which the elements are u_0, u_1, $u_2 \ldots$ subject to the above relation. An infinite sorites read in the limiting case is therefore expressible under the form of a finite fraction, and the same will be true for a sorites in which the right-hand branch beginning from any term u_i, namely, $\frac{1}{u_i} + \frac{1}{u_{i+1}} + \frac{1}{u_{i+2}} \ldots$, forms a limiting sorites.

But in every other case of a sorites the sum cannot be a finite fraction; for such fraction can be expanded in only one way under the form of a sorites, and such sorites is necessarily finite in the number of its terms.

Hence it is impossible that the sum of the reciprocals of an ascending series of positive integers, such that the square root of the difference between any one of them and its immediate antecedent is greater than the difference between that antecedent and unity, can represent a rational quantity; for if so, we have $u_{x+1} - u_x$ greater than $(u_{x-1} - 1)^2$, i. e. $u_{x+1} > u_x^2 - u_x + 1$, and the series will form a sorites not belonging to the limiting class.

I proceed to examine some of the properties of the series of terms defined by the condition $u_{x+1} = u_x^2 - u_x + 1$.

In the first place, I observe that any term u_{x+i} may be expressed under the form $Pu_x + 1$: for suppose this to be true for one value of i; then, since $u_{x+i+1} - 1 = u_{x+i}(u_{x+i} - 1)$, it is obviously true for the next above; here the proposition, being true when i is unity, is true universally.

It follows from this that each element of a limiting sorites is prime to all that follow it, and consequently any two terms of the sorites are prime to one another.

Again, for greater simplicity, let $v_0, v_1, v_2, \ldots$ be used to represent $(u_0 - 1)$, $(u_1 - 1)$, $(u_2 - 1)$, ...; we have, then,

$$v_1 - v_0 = v_0^2, \qquad v_2 - v_1 = v_1^2, \qquad v_3 - v_2 = v_2^2, \ldots$$

Hence $v_2 - v_0, v_3 - v_0, \ldots . v_x - v_0$ (as is obvious from successive addition of the above equations) will each of them be of the form Pv_0^2, where P is a rational integral function of v_0, and v_x will be of the form $Pv_0^2 + v_0$. This conclusion leads to a representation of the sum of any given number of terms of a limiting sorites by a fraction in its lowest terms. For

$$\frac{1}{v_x} - \frac{1}{v_{x+1}} = \frac{v_{x+1} - v_x}{v_x v_{x+1}} = \frac{v_x^2}{v_x v_{x+1}} = \frac{v_x}{v_x + v_x^2} = \frac{1}{v_x + 1} = \frac{1}{u_x}.$$

Hence

$$\frac{1}{u_0} + \frac{1}{u_1} + \ldots . + \frac{1}{u_x} = \frac{1}{v_0} - \frac{1}{v_{x+1}} = \frac{v_{x+1} - v_0}{v_0 v_{x+1}} = \frac{(v_{x+1} - v_0) \div v_0^2}{v_{x+1} \div v_0},$$

which is of the form $\frac{P}{Pv_0 + 1}$ and is consequently a fraction in its lowest terms.

Again, if we denote the product of the elements $u_0, u_1, u_2, \ldots . , u_x$ by Πu_x and the sum of their $(x-1)$-ary combinations by $\Pi' u_x$, $\frac{\Pi' u_x}{\Pi u_x}$ will also be the same fraction in its lowest terms, because (as has been shown) all the elements of the sorites are prime to one another.

Hence we may deduce the equations

$$u_{x+1} = u_0 + (u_0 - 1)^2 \, \Pi' u_x,$$

$$u_{x+1} = 1 + (u_0 - 1) \, \Pi u_x.$$

The second of these serves to give an inferior limit to the rate of convergence of any sorites. For in the limiting case we have

$$u_1 > (u_0^2 - u_0),$$

$$u_2 > (u_0 - 1) \, u_0 u_1 > (u_0^2 - u_0)^2,$$

$$u_3 > (u_0 - 1) \, u_0 u_1 u_2 > (u_0^2 - u_0)^4,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

and so in general $u_x > (u_0^2 - u_0)^{2^{x-1}}$, because the solution of the equation $\theta_i = \theta_{i-1} + \theta_{i-2} + \ldots . + \theta_0$ is $\theta_i = 2^{i-1}\theta_0$. In any other sorites in which the initial element remains u_0, the value of the element at x-places distant must be *à fortiori* greater than the value $(u_0^2 - u_0)^{2^{x-1}}$ last obtained for the limiting case.

The preceding matter was suggested to me by the chapter in Cantor's *Geschichte der Mathematik* which gives an account of the singular method in use among the ancient Egyptians for working with fractions. It was their curious custom to resolve every fraction into a sum of simple fractions according to a certain traditional method, not leading, I need hardly say, except in a few of the

simplest cases, to the expansion under the special form to which I have, in what precedes, given the name of a fractional *sorites*.

I subjoin examples of development of a rational fraction under the form of a sorites.

Let $\frac{4699}{7320}$ be the fraction to be expanded. The work may be arranged as follows:—

(2)	(8)	(60)	(3660)
4699	2078	1984	1920
7320	14640	117120	7027200
9398	16624	119040	7027200

(2) is the number one unit greater than $E\,\frac{7320}{4699}$; 9398 is 2 × 4699; 2078 is 9398 − 7320; 14640 is 2 × 7320.

One element (2) is now determined, and the fraction $\frac{2078}{14640}$ remains to be expanded.

(8) is the number one unit greater than $E\,\frac{14640}{2078}$; 16624 is 8 × 2078; 1984 is 16624 − 14640; 117120 is 8 × 14640.

A second element (8) is now found, and $\frac{1984}{117120}$ remains over to be expanded. Proceeding in this manner, and with numerators 4699, 2078, 1984, 1920, necessarily diminishing at each step, we come at last to the element 3660 with a remainder zero. The required sorites is therefore

$$\frac{1}{2}+\frac{1}{8}+\frac{1}{60}+\frac{1}{3660}.$$

As a second example take the fraction $\frac{335}{336}$.

The work may be arranged in a similar manner to that of the foregoing example, and will be as follows:—

2	3	7	48
335	334	330	294
336	672	2016	14112
670	1002	2310	14112

and accordingly it will be found that

$$\frac{335}{336}=\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{48}.$$

On an Immediate Generalization of Local Theorems in which the Generating Point divides a Variable Linear Segment in a Constant Ratio.

BY SAMUEL ROBERTS,

President of the London Mathematical Society.

1. IN what follows I make use of the obvious principle, that, if two curves have a one to one correspondence, and if the points on a particular straight line of one correspond to points on a straight line of the other, the curves are of the same order and deficiency. The particular straight line is, in the cases I shall discuss, the line at infinity for both curves.

2. A curve may be determined as the locus of a point, which divides in a constant ratio a terminated straight line variable in length and position; and a family of curves related to one another is obtained by changing the ratio. Each such family depends, therefore, on one parameter. An additional parameter is introduced by the transformation of which I propose to treat.

There are commonly particular values of the ratio, which give curves of a lower order than that of the general locus, especially by the reduplication of the locus or a curve-factor of the locus.

For example, if straight lines through a point meet a circle, the locus of the middle points of the intercepts by the circle is another circle through the fixed point, and the centre of the direction.

This is plainly a special locus, and if the chords are divided by the generating point in the ratio $k : l$, the locus is found to be the inverse of a conic, i. e. of the fourth order. We may look for a similar reduction of the order whenever the ratio is one of equality, and the extremities of the linear segments move on one and the same curve.

It is to be understood that when I hereafter speak of a curve of the class described, which for brevity I shall call a ratio-curve, I mean one general as to order; in fact, I suppose the ratio of division to be denoted by general literal symbols, to which such values may be ascribed as will not give rise to a special reduction of order.

3. I propose to establish, that, if similar triangles, uniformly directed with respect to the generating segments, be superposed thereon, each having for base the corresponding segment, the general locus of the vertex, say, the vertex-curve, is of the same order and the same deficiency as the corresponding ratio-curve.

This is known to be true, for instance, of loci so derived from a constant line moving in plane space. The triangles are then all equal, and the result is, that the general locus of a point rigidly connected with the moving line is of the same order and deficiency as the corresponding ratio-curve,—a conclusion of some kinematical importance.

The statement is also manifestly true with respect to a curve referred to polar co-ordinates. If we divide the radii vectores in a given ratio, a similar and similarly placed curve is described by the dividing point; and if triangles are constructed on these vectores in the manner proposed, we have a similar curve turned through an angle.

And first of all, as to the deficiency, the correspondence of the ratio-curve and the vertex-curve is, by the nature of the construction, one to one, so that their deficiency is the same. The complete ratio and vertex curves may, however, break up into corresponding factors, and if so, the deficiencies of the corresponding factors are the same. It is the question of order which requires special consideration.

4. Let AB be the linear segment in one of its positions, divided at the point P in a given ratio. Erect QP perpendicular to AB. The triangle AQB will remain similar and uniformly directed if

$$AP : BP : PQ = k : l : p,$$

where k, l, p are constant and finite quantities.

If, therefore, X, Y, are the co-ordinates of Q; x_1, y_1, and x_2, y_2 those of A and B, respectively; and θ is the angle which AB makes with a fixed line, the axis of x, we have

$$\left.\begin{aligned} X &= x_1 + m\,(k \cos\theta - p \sin\theta) \\ Y &= y_1 + m\,(k \sin\theta + p \cos\theta) \\ X &= x_2 + m\,(l \cos\theta - p \sin\theta) \\ Y &= y_2 + m\,(l \sin\theta + p \cos\theta), \end{aligned}\right\} \quad \text{(A)}$$

where m and θ are variable parameters.

In order to define the locus, we must have further conditions equivalent to three independent conditions in $x_1, y_1, x_2, y_2, m, \theta$. These may be taken to be independent of k, l, p; for if these constants should be included among the constants of the complementary conditions, we are at liberty to change p, k, l in (A) into p', k', l'.

Now, although by giving form to such further conditions, we can represent cases of great generality, the complete proposition cannot be established by this means. We must obtain our conclusion independently of the unexpressed conditions.

The system of equations corresponding to (A), but belonging to the ratio-curve described by the foot of the perpendicular QP, is

$$\left.\begin{aligned} X' &= x_1 + mk\cos\theta \\ Y' &= y_1 + mk\sin\theta \\ X' &= x_2 + ml\cos\theta \\ Y' &= y_2 + ml\sin\theta. \end{aligned}\right\} \quad \text{(B)}$$

By combining (A) and (B), we may obtain a variety of relations. It will be sufficient to write down

$$\begin{aligned} (l-k)\,X' &= lx_1 - kx_2 \\ (l-k)\,Y' &= ly_1 - ky_2 \end{aligned} \quad \text{(a)}$$

$$\begin{aligned} X &= X' - \tfrac{p}{k}(Y' - y_1) \qquad & X &= X' - \tfrac{p}{l}(Y' - y_2) \\ Y &= Y' + \tfrac{p}{k}(X' - x_1) \qquad & Y &= Y' + \tfrac{p}{l}(X' - x_2). \end{aligned} \quad \text{(b)}$$

5. Now, if we write for x_1, $\rho_1(\cos\alpha + i\sin\alpha)$, for y_1, $r_1(\cos\beta + i\sin\beta)$, etc., where α, β, etc., are real angles and ρ_1, r_1, etc., are real and positive moduli, we see, from the foregoing expressions, that, if the moduli of (x_1, y_1), (x_2, y_2) are finite, then those of (X', Y') and (X, Y) are also finite. And if the moduli of (x_1, y_1) or of (x_2, y_2) are infinite, or one of them is infinite, but the moduli of the remaining co-ordinates (x_2, y_2) or (x_1, y_1) are finite, then the moduli of (X', Y') and (X, Y), or one of each is infinite.*

No doubt we ought to be able to derive all the conclusions we want from the same systems (A) and (B), when the segment lies altogether at infinity. There

* This is so stated to cover the case of a point at infinity, determined by a line parallel to an axis, when one of the co-ordinates is finite.

are, however, obvious difficulties in the way, especially when we consider the circular asymptotes. These are "lines of no length." Nevertheless, the co-ordinates of points on a circular asymptote have determinate moduli, so that part of the difficulty may be evaded; but, again, a circular asymptote makes any real angle with itself. It is more satisfactory to employ another process than to argue directly on infinites and the circular points.

6. By the very nature of the construction the correspondence of (X', Y') and (X, Y) depends on the position and length of the corresponding segment alone, and to determine it, we are at liberty to arrive at any proposed position of the segment as we conveniently can. Moreover, the actual length of the segment is immaterial when we have only to distinguish between finite and infinitely distant points. This applies to imaginary as well as real positions. These considerations enable us to have recourse to certain elementary cases; and to avoid direct reasoning on isolated infinite quantities, I take, therefore, some essential cases in order.

I. *One extremity of the segment finite and real, the other at infinity and real.*

Let A be the finite extremity. In the direction of the segment take AB finite, and construct the triangle AQB and the perpendicular QP, as before.

If now we suppose B to move to infinity, the triangle AQB remaining of the same angularity, P and Q ultimately lie at infinity, P coinciding with B, and Q being determined by the direction AQ.

This agrees with the result of neglecting x_2, y_2 in (a) and (b).

II. *The segment real and altogether at infinity.*

Take the lines OA, OB met by the finite segment AB, and construct the triangle AQB and the perpendicular QP, as before.

When AB moves to infinity, parallel to itself, P and Q also move to infinity, when the angularity of AQB is unchanged.

Since, however, the direction of AB is arbitrary, every point at infinity in turn may be made to represent the ultimate position of P, indicating that the line at infinity becomes an extraneous factor of the locus, the order of which is consequently reduced. But if we determine the direction of P, that of Q is also determined for a given angularity. The point to be noticed is that the mutually corresponding points, whether proper or extraneous, all lie at infinity.

The conclusions (I.) and (II.) will hold for imaginary positions of the segment when we have regard to the moduli of the co-ordinates, except that the case in which the segment coincides with a circular asymptote requires special determination.

7. III. Therefore, consider the elementary case of straight lines through a point and meeting two given straight lines. The ratio and vertex curves are conics, passing through the points at infinity, corresponding respectively to those of the directrices. And since in this case all finite points (X', Y') correspond to finite points (X, Y), and *vice versa*, it follows, generally, that this result holds for finite positions of the segment, whether imaginary or real, even when the segment forms part of a circular asymptote. But since this is so, and the correspondence of the ratio-curve to the vertex-curve is one to one, it follows that to infinitely distant points on the ratio-curve must correspond infinitely distant points on the vertex-curve. Consequently the order must be the same for both.

For the purpose of showing the effect when the one extremity of the segment coincides with a circular point at infinity, it is more convenient to take the case of a point and circle.

The ratio-curve and the vertex-curve are both of them circles, and the corresponding points (X', Y') (X, Y) coincide at the circular points.

IV. We must consider the case in which the segment passes through both the circular points, and therefore coincides with the line at infinity.

If parallel lines meet a straight line and a circle, the ratio and vertex curves are conics, and the infinitely distant points of one conic correspond to those, respectively, of the other.

According to the same principles, the effect may be shown of the coincidence of two extremities of a segment. Thus, if a pencil of parallel lines meets a parabola, the ratio-curve and the vertex-curve are both parabolas, and the points of contact with the line at infinity correspond. It is not necessary, however, to pursue this further for the end in view.

We see, then, that to points at infinity on the ratio-curve always correspond points at infinity on the vertex-curve; and since the correspondence is one to one, the order must be the same for both.

Since the vertex-curve can be obtained by taking the ratio-curve as one of the directrices, to an extraneous factor of the one will correspond an extraneous factor of the other; and if the one curve is composite, so is the other in a corresponding manner.

It is to be observed that such composite loci may still be reckoned as proper loci, in our present point of view.

8. The conclusion at which we have arrived is of considerable utility for the extension of problems on loci. In many cases a ratio-curve can be determined by simple processes, when the analytical investigation of the corresponding vertex-curve involves complicated work.

Take, for instance, a pencil of straight lines meeting two directrices. In the ratio-curve, adopting polar co-ordinates with the vertex of the pencil as pole, we have to eliminate ρ_1, ρ_2 from

$$R = m\rho_1 + n\rho_2$$

$$\phi(\rho_1, \theta) = 0 \qquad \psi(\rho_2, \theta) = 0.$$

This is often a simple matter where the corresponding elimination for the vertex-curve is lengthy or impracticable.

Suppose the directrices are two circles, the ratio-curve is readily found to be of the sixth order (tricircular), with three other nodes, reducing to the fourth order (the inverse of a conic) when the pole is on one of the circles, and to the second order (a circle) when the pole is one of the finite intersections of the directrices. It is tedious, though of course practicable, to show analytically that the vertex-curve is also in the first case a sextic of similar kind, in the second the inverse of a conic, and in the third a circle. The last result is, however, well known in connection with the theory of three circles intersecting in a point.

9. Again, it will also be observed that we can project orthogonally a figure relative to a ratio-curve, and reproduce a ratio-curve together with its appropriate conditions of generation. The vertex-curve cannot be so projected, and consequently we get an entirely new theorem.

For example, projecting the last case, above given, we have the following:—

If a vector be drawn through a finite intersection of two similar and similarly placed conics, the locus of a point dividing the intercepts in a constant ratio is a similar and similarly placed conic passing through the finite intersections together, strictly speaking, with two similar and similarly placed conics obtained by measuring the intercepts from the pole.

We have for the vertex-curve the following:—

If a vector be drawn through a finite intersection of two similar and similarly placed conics, the vertices of similar and similarly directed triangles on the intercepts as base describe a conic through the other finite intersection of the two given conics (a conic not necessarily similar or similarly placed), together with two supplementary conics. This result is not to be derived by projection, since the triangles would not be projected into similar triangles.

10. It has served our purpose to consider examples belonging to the very simple case of lines enveloping a point. To illustrate different conditions, suppose that straight lines touch a given circle and meet a given straight line; the

locus of a point dividing the intercepts on the tangents between the point of contact and the line is a curve of the fourth order. In this case we get

$$X = x_1 + m\,(k \cos\theta - p \sin\theta)$$

$$Y = y_1 + m\,(k \sin\theta + p \cos\theta)$$

$$y_1 - \delta = 0$$

$$X = x_2 + m\,(l \cos\theta - p \sin\theta)$$

$$Y = y_2 + m\,(l \sin\theta + p \cos\theta)$$

$$x_2 \cos\theta + y_2 \sin\theta = 0$$

$$x_2^2 + y_2^2 = r^2,$$

whence we have

$$X \cos\theta + Y \sin\theta = ml$$

$$X \sin\theta - Y \cos\theta = -mp \pm r$$

$$Y - \delta = m\,(k \sin\theta + p \cos\theta).$$

Simple as these equations are, the reduction is troublesome; but if $p = 0$, then

$$X^2 + Y^2 = m^2 l^2 + r^2$$

$$mkX \sin\theta = X\,(Y - \delta)$$

$$mkX \cos\theta = m^2 kl - Y\,(Y - \delta),$$

giving

$$\frac{k^2}{l^2} X^2 (X^2 + Y^2 - r^2) - X^2 (Y - \delta)^2 - \left(\frac{k}{l}(X^2 + Y^2 - r^2) - Y(Y - \delta)\right)^2 = 0.$$

The curve is circular, having a double point at infinity determined by $Y = 0$, in accordance with our previous conclusions.

The vertex-curve is, therefore, also circular, of the fourth order, having a double point at infinity determined by $Xp - Yk = 0$.

We can now project the circle and line directrices into a central conic and line, and deduce a similar result for the vertex-curve in this case.

11. I have only further to remark, that the transformation of the ratio-curve into the vertex-curve is a particular case of the general transformation where the powers and products of X, Y are decomposed into their elements, and each X and Y receives an independent linear transformation.

We see this directly by writing (A) in the form

$$Xk + Yp = x_1k + y_1p + m\,(p^2 + k^2)\cos\theta = X_1 + m\,(p^2 + k^2)\cos\theta$$
$$Yk - Xp = y_1k - x_1p + m\,(p^2 + k^2)\sin\theta = Y_1 + m\,(p^2 + k^2)\sin\theta$$
$$Xl + Yp = x_2l + y_2p + m\,(p^2 + l^2)\cos\theta = X_2 + m\,(p^2 + l^2)\cos\theta$$
$$Yl - Xp = y_2l - x_2p + m\,(p^2 + l^2)\sin\theta = Y_2 + m\,(p^2 + l^2)\sin\theta$$

and

$$x_1 = \frac{X_1k - Y_1p}{p^2 + k^2} \qquad y_1 = \frac{X_1p + Y_1k}{p^2 + k^2}$$
$$x_2 = \frac{X_2l - Y_2p}{p^2 + l^2} \qquad y_2 = \frac{X_2p + Y_2l}{p^2 + l^2},$$

and by comparing these with (B).

Since, however, in this way of putting the matter, the unexpressed conditions vary with p, k, l, we cannot at once infer the equality of the orders of the ratio and vertex curves, for which purpose it has been necessary to enter into somewhat tedious details. I have not been able to hit upon a simpler proof, though it is highly probable such a one can be found. I thought it desirable to introduce the analytical expressions of Article 4, in order to show where the difficulty arises, but in fact we found them insufficient, and it would favor uniformity to adhere to the method of Articles 6 and 7.

On the Expansion of $\phi(x+h)$.

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The object of this paper is the development of Taylor's formula, the development of the form of the functional coefficient of h (usually known as θ), in the remainder in that formula, and the development of the forms of $\theta_1, \theta_2 \ldots.$ as they appear in the equations $(a)', (b)' \ldots.$

Suppose ϕx and $\phi' x$ to be finite and continuous functions for all values of x from $x = x'$ to $x = x' + mh$, m being always positive, and either a constant, or a function of x and h of such form as to reduce to a constant when $h = 0$. Then will

$$\frac{\phi(x' + mh) - \phi x'}{mh} = \phi'(x' + \theta' h),$$

where θ' is between 0 and m.

Let A and B be the algebraically greatest and least values of $\phi' x$ for values of x between x' and $x' + mh$.

Put

$$y = Ax - \phi x, \tag{1}$$

and

$$z = \phi x - Bx. \tag{2}$$

Then

$$\frac{dy}{dx} = A - \phi' x,$$

and

$$\frac{dz}{dx} = \phi' x - B,$$

both of which are positive for all values of x between x' and $x' + mh$, and consequently y and z are increasing functions for all values of x between x' and $x' + mh$.

Let x take each of the two values x' and $x'+mh$ in both (1) and (2). Then

$$y'' = A(x'+mh) - \phi(x'+mh),$$

$$z'' = \phi(x'+mh) - B(x+mh),$$

$$y' = Ax' - \phi x',$$

and

$$z' = \phi x - Bx'.$$

Then

$$\frac{y''-y'}{mh} = A - \frac{\phi(x'+mh) - \phi x'}{mh}, \quad \text{and} \quad \frac{z''-z'}{mh} = \frac{\phi(x'+mh) - \phi x'}{mh} - B.$$

Because y and z are increasing functions, and m is positive, $y''-y'$ and $z''-z'$ are of the same sign as h, and the members of the last two equations are positive. Consequently $\frac{\phi(x'+mh)-\phi x'}{mh}$ is less than A and greater than B, and therefore, by reason of the continuity of $\phi' x$ between A and B, is equal to some value of $\phi' x$ between A and B in which x has some value between x' and $x'+mh$. Let $x'+\theta' h$ represent this value of x, θ' being between 0 and m. Then

$$\frac{\phi(x'+mh)-\phi x'}{mh} = \phi'(x'+\theta' h),$$

or

$$\phi(x+h) = \phi x + mh\phi'(x+\theta' h), \tag{3}$$

which will hold true for all values of x and mh within the limits of those giving finite and continuous values in ϕx and $\phi' x$.

Differentiating (3) twice, regarding x as constant, and in the result putting $h=0$,

$$(m^2)_{h=0}\phi'' x = 2(m\theta')_{h=0}\phi'' x.$$

$$\therefore \ (\theta')_{h=0} = \frac{1}{2}(m)_{h=0}, \quad \text{and} \quad \theta' = \frac{1}{2}(m)_{h=0} + \theta^0,$$

where $\theta^0 = 0$ when $h=0$. These results obtain whether m be constant or variable.

Since θ' is always between 0 and m, and m is always positive, and since θ' reduces to the constant $\frac{1}{2}(m)_{h=0}$ when $h=0$, θ', like m, is always positive, and either a constant, or reduces to one when $h=0$. Hence, if $\phi'' x$, $\phi''' x \ldots \phi^{2n} x$ are finite and continuous functions the same as ϕx and $\phi' x$, we can form the

following equations, in which θ'', θ''' .... are, like m and θ', always **positive, and** either constants, or reduced to constants when $h = 0$.

$$\phi(x+mh) = \phi x + mh\phi'(x+\theta'h), \qquad (a)$$

where θ' is between 0 and m;

$$\phi'(x+\theta'h) = \phi'x + \theta'h\phi''(x+\theta''h), \qquad (b)$$

where θ'' is between 0 and θ';

$$\phi''(x+\theta''h) = \phi''x + \theta''h\phi'''(x+\theta'''h), \qquad (c)$$

where θ''' is between 0 and θ'';

. . . .

$$\phi^{2n-1}(x+\theta^{2n-1}h) = \phi^{2n-1}x + \theta^{2n-1}h\phi^{2n}(x+\theta^{2n}h), \qquad (d)$$

where θ^{2n} is between 0 and θ^{2n-1}.

The same law that makes

$$(\theta')_{h=0} = \frac{1}{2}(m)_{h=0},$$

also makes

$$(\theta'')_{h=0} = \frac{1}{2}(\theta')_{h=0} = \frac{1}{2^2}(m)_{h=0},$$

$$(\theta''')_{h=0} = \frac{1}{2}(\theta'')_{h=0} = \frac{1}{2^3}(m)_{h=0},$$

and in general,

$$(\theta^{2n})_{h=0} = \frac{1}{2^{2n}}(m)_{h=0}.$$

Let $m = 1$, and denote what θ', θ'', θ'''.... become by θ_1, θ_2, θ_3.... **Then,**

$$\phi(x+h) = \phi x + h\phi'(x+\theta_1 h), \qquad (a)$$

where θ_1 is between 0 and 1;

$$\phi'(x+\theta_1 h) = \phi'x + \theta_1 h\phi''(x+\theta_2 h), \qquad (b)$$

where θ_2 is between 0 and θ_1;

$$\phi''(x+\theta_2 h) = \phi''x + \theta_2 h\phi'''x(x+\theta_3 h), \qquad (c)$$

where θ_3 is between 0 and θ_2;

. . . .

$$\phi^{2n-1}(x+\theta_{2n-1}h) = \phi^{2n-1}x + \theta_{2n-1}h\phi^{2n}(x+\theta_{2n}h), \qquad (d)$$

where θ_{2n} is between 0 and θ_{2n-1}.

Further, if $h = 0$, then

$$(\theta_1)_{h=0} = \frac{1}{2}, \qquad (\theta_2)_{h=0} = \frac{1}{4},$$

$$(\theta_3)_{h=0} = \frac{1}{8} \dots (\theta_{2n})_{h=0} = \frac{1}{2^{2n}}.$$

Forming the successive diff. co. of (3), regarding x as constant, and in the results putting $h = 0$,

$$(m)_{h=0}\phi' x = (m)_{h=0}\phi' x. \tag{4}$$

$$(m^2)_{h=0}\phi'' x = 2(m\theta')_{h=0}\phi'' x. \tag{5}$$

$$6\left(m\frac{dm}{dh}\right)_{h=0}\phi'' x + (m^3)_{h=0}\phi''' x = 6\left(\theta'\frac{dm}{dh}\right)_{h=0}\phi'' x + 6\left(m\frac{d\theta'}{dh}\right)_{h=0}\phi'' x + 3(m\theta'^2)_{h=0}\phi''' x. \tag{6}$$

$$\begin{aligned}12\left(m\frac{d^2m}{dh^2}\right)_{h=0}\phi'' x + 12\left(\frac{dm}{dh}\right)^2_{h=0}\phi'' x + 12\left(m^2\frac{dm}{dh}\right)_{h=0}\phi''' x + (m^4)_{h=0}\phi^{\mathrm{IV}} x = 12\left(\theta'\frac{d^2m}{dh^2}\right)_{h=0}\phi'' x \\ + 24\left(\frac{dm}{dh}\frac{d\theta'}{dh}\right)_{h=0}\phi'' x + 12\left(\theta'^2\frac{dm}{dh}\right)_{h=0}\phi''' x + 12\left(m\frac{d^2\theta'}{dh^2}\right)_{h=0}\phi'' x + 24\left(m\theta'\frac{d\theta'}{dh}\right)_{h=0}\phi''' x \\ + 4(m\theta'^3)_{h=0}\phi^{\mathrm{IV}} x.\end{aligned} \tag{7}$$

$$\begin{aligned}20\left(m\frac{d^3m}{dh^3}\right)_{h=0}\phi'' x + 60\left(\frac{dm}{dh}\frac{d^2m}{dh^2}\right)_{h=0}\phi'' x + 30\left(m^2\frac{d^2m}{dh^2}\right)_{h=0}\phi''' x + 60\left[m\left(\frac{dm}{dh}\right)^2\right]_{h=0}\phi''' x \\ + 20\left(m^3\frac{dm}{dh}\right)_{h=0}\phi^{\mathrm{IV}} x + (m^5)_{h=0}\phi^{\mathrm{V}} x = 20\left(\theta'\frac{d^3m}{dh^3}\right)_{h=0}\phi'' x + 60\left(\frac{d^2m}{dh^2}\frac{d\theta'}{dh}\right)_{h=0}\phi'' x \\ + 30\left(\theta'^2\frac{d^2m}{dh^2}\right)_{h=0}\phi''' x + 60\left[m\left(\frac{d\theta'}{dh}\right)^2\right]_{h=0}\phi''' x + 20\left(\theta'^3\frac{dm}{dh}\right)_{h=0}\phi^{\mathrm{IV}} x + 5(m\theta'^4)_{h=0}\phi^{\mathrm{V}} x \\ + 60\left(\frac{dm}{dh}\frac{d^2\theta'}{dh^2}\right)_{h=0}\phi'' x + 120\left(\theta'\frac{dm}{dh}\frac{d\theta'}{dh}\right)_{h=0}\phi''' x + 20\left(m\frac{d^3\theta'}{dh^3}\right)_{h=0}\phi'' x + 60\left(m\theta'\frac{d^2\theta'}{dh^2}\right)_{h=0}\phi''' x \\ + 60\left(m\theta'^2\frac{d\theta'}{dh}\right)_{h=0}\phi^{\mathrm{IV}} x.\end{aligned} \tag{8}$$

$$\begin{aligned}30\left(m\frac{d^4m}{dh^4}\right)_{h=0}\phi'' x + 120\left(\frac{dm}{dh}\frac{d^3m}{dh^3}\right)_{h=0}\phi'' x + 60\left(m^2\frac{d^3m}{dh^3}\right)_{h=0}\phi''' x + 90\left(\frac{d^2m}{dh^2}\right)^2_{h=0}\phi'' x \\ + 360\left(m\frac{dm}{dh}\frac{d^2m}{dh^2}\right)_{h=0}\phi''' x + 60\left(m^2\frac{d^2m}{dh^2}\right)_{h=0}\phi^{\mathrm{IV}} x + 120\left(\frac{dm}{dh}\right)^3_{h=0}\phi''' x \\ + 180\left[m^2\left(\frac{dm}{dh}\right)^2\right]_{h=0}\phi^{\mathrm{IV}} x + 30\left(m^4\frac{dm}{dh}\right)_{h=0}\phi^{\mathrm{V}} x + (m^6)_{h=0}\phi^{\mathrm{VI}} x = 30\left(\theta'\frac{d^4m}{dh^4}\right)_{h=0}\phi'' x \\ + 120\left(\frac{d^3m}{dh^3}\frac{d\theta'}{dh}\right)_{h=0}\phi'' x + 60\left(\theta'^2\frac{d^3m}{dh^3}\right)_{h=0}\phi''' x + 180\left(\frac{d^2m}{dh^2}\frac{d^2\theta'}{dh^2}\right)_{h=0}\phi'' x\end{aligned}$$

$$+ 360\left(\theta' \frac{d^2m}{dh^2}\frac{d\theta'}{dh}\right)_{h=0}\phi'''x + 60\left(\theta'^3 \frac{d^2m}{dh^2}\right)_{h=0}\phi^{\mathrm{IV}}x + 120\left(\frac{dm}{dh}\frac{d^3\theta'}{dh^3}\right)_{h=0}\phi''x$$
$$+ 360\left(\theta' \frac{dm}{dh}\frac{d^2\theta'}{dh^2}\right)_{h=0}\phi'''x + 360\left[\frac{dm}{dh}\left(\frac{d\theta'}{dh}\right)^2\right]_{h=0}\phi'''x + 360\left(\theta'^2 \frac{dm}{dh}\frac{d\theta'}{dh}\right)_{h=0}\phi^{\mathrm{IV}}x$$
$$+ 30\left(m\frac{d^4\theta'}{dh^4}\right)_{h=0}\phi''x + 120\left(m\theta'\frac{d^3\theta'}{dh^3}\right)_{h=0}\phi'''x + 360\left(m\frac{d\theta'}{dh}\frac{d^2\theta'}{dh^2}\right)_{h=0}\phi'''x$$
$$+ 180\left(m\theta'^2\frac{d^2\theta'}{dh^2}\right)_{h=0}\phi^{\mathrm{IV}}x + 30\left(\theta'^4\frac{dm}{dh}\right)_{h=0}\phi^{\mathrm{V}}x + 360\left[m\theta'\left(\frac{d\theta'}{dh}\right)^2\right]_{h=0}\phi^{\mathrm{IV}}x$$
$$+ 120\left(m\theta'^3\frac{d\theta'}{dh}\right)_{h=0}\phi^{\mathrm{V}}x + 6\,(m\theta'^5)_{h=0}\,\phi^{\mathrm{VI}}x. \qquad (9)$$

. . . .

If in (4), (5), (6), (7), (8), and (9) we put $m = 1$, and write θ_1 for θ', we obtain the same results as when $h = 0$ in the successive diff. co., with respect to h, of $(a)'$. Hence,

$$\phi' x = \phi' x. \qquad (10)$$

$$\phi'' x = 2\,(\theta_1)_{h=0}\phi'' x. \qquad (11)$$

$$\phi''' x = 6\left(\frac{d\theta_1}{dh}\right)_{h=0}\phi'' x + 3\,(\theta_1^2)_{h=0}\,\phi''' x. \qquad (12)$$

$$\phi^{\mathrm{IV}} x = 12\left(\frac{d^2\theta_1}{dh^2}\right)_{h=0}\phi'' x + 24\left(\theta_1\frac{d\theta_1}{dh}\right)_{h=0}\phi''' x + 4\,(\theta_1^3)_{h=0}\phi^{\mathrm{IV}} x. \qquad (13)$$

$$\phi^{\mathrm{V}} x = 20\left(\frac{d^3\theta_1}{dh^3}\right)_{h=0}\phi'' x + 60\left(\theta_1\frac{d^2\theta_1}{dh^2}\right)_{h=0}\phi''' x + 60\left(\frac{d\theta_1}{dh}\right)^2_{h=0}\phi''' x + 60\left(\theta_1^2\frac{d\theta_1}{dh}\right)_{h=0}\phi^{\mathrm{IV}} x$$
$$+ 5\,(\theta_1^4)_{h=0}\phi^{\mathrm{V}} x. \qquad (14)$$

$$\phi^{\mathrm{VI}} x = 30\left(\frac{d^4\theta_1}{dh^4}\right)_{h=0}\phi'' x + 120\left(\theta_1\frac{d^3\theta_1}{dh^3}\right)_{h=0}\phi''' x + 360\left(\frac{d\theta_1}{dh}\frac{d^2\theta_1}{dh^2}\right)_{h=0}\phi''' x + 180\left(\theta_1^2\frac{d^2\theta_1}{dh^2}\right)_{h=0}\phi^{\mathrm{IV}} x$$
$$+ 360\left[\theta_1\left(\frac{d\theta_1}{dh}\right)^2\right]_{h=0}\phi^{\mathrm{IV}} x + 120\left(\theta_1^3\frac{d\theta_1}{dh}\right)_{h=0}\phi^{\mathrm{V}} x + 6\,(\theta_1^5)_{h=0}\phi^{\mathrm{VI}} x. \qquad (15)$$

Similarly,

$$\phi^{\mathrm{VII}} x = 42\left(\frac{d^5\theta_1}{dh^5}\right)_{h=0}\phi'' x + 210\left(\theta_1\frac{d^4\theta_1}{dh^4}\right)_{h=0}\phi''' x + 840\left(\frac{d\theta_1}{dh}\frac{d^3\theta_1}{dh^3}\right)_{h=0}\phi''' x + 420\left(\theta_1^2\frac{d^3\theta_1}{dh^3}\right)_{h=0}\phi^{\mathrm{IV}} x$$
$$+ 630\left(\frac{d^2\theta_1}{dh^2}\right)^2_{h=0}\phi'' x + 2520\left(\theta_1\frac{d\theta_1}{dh}\frac{d^2\theta_1}{dh^2}\right)_{h=0}\phi^{\mathrm{IV}} x + 840\left(\frac{d\theta_1}{dh}\right)^3\phi^{\mathrm{IV}} x$$
$$+ 1260\left[\theta_1^2\left(\frac{d\theta_1}{dh}\right)^2\right]_{h=0}\phi^{\mathrm{V}} x + 420\left(\theta_1^3\frac{d^2\theta_1}{dh^2}\right)_{h=0}\phi^{\mathrm{V}} x + 210\left(\theta_1^4\frac{d\theta_1}{dh}\right)_{h=0}\phi^{\mathrm{VI}} x$$
$$+ 7\,(\theta_1^6)_{h=0}\phi^{\mathrm{VII}} x. \qquad (16)$$

. . . .

Equation $(a)'$ reduces to $\phi x = \phi x$, when $h = 0$, but its diff. co., with respect to h, reduces to $\phi' x = \phi' x$, by (10). Hence by one differentiation with respect to h, h has been eliminated from one term in $h\phi'(x + \theta_1 h)$. Consequently one term in $h\phi'(x + \theta_1 h)$ is $h\phi' x$.

Again: when $h = 0$, the second diff. co. of $(a)'$ with respect to h, reduces to $\phi'' x = 2(\theta_1)_{h=0}\phi'' x$, by (11). Consequently by two differentiations with respect to h, $\frac{h^2}{2}$ has been eliminated from a second term in $h\phi'(x + \theta_1 h)$, and a second term in that expression is

$$\frac{h^2}{2}\phi'' x = \frac{h^2}{2}\left[2(\theta_1)_{h=0}\phi'' x\right].$$

In like manner by three differentiations with respect to h, and in the result putting $h = 0$, (12) is obtained from $(a)'$, where it is plain $\frac{h^3}{2.3}$ has been eliminated from a third term in $h\phi'(x + \theta_1 h)$, and a third term in in that expression is

$$\frac{h^3}{2.3}\phi''' x = \frac{h^3}{2.3}\left[6\left(\frac{d\theta_1}{dh}\right)_{h=0}\phi'' x + 3(\theta_1^2)_{h=0}\phi''' x\right].$$

Additional terms may be found in the same way from (13), (14) ...., and with the foregoing ones substituted for $h\phi'(x + \theta_1 h)$ in $(a)'$, giving

$$\phi(x+h) = \phi x + h\phi' x + \frac{h^2}{2}\phi'' x + \frac{h_3}{2.3}\phi''' x + \ldots . \frac{h^n}{\lfloor n}\phi^n x + \ldots . \tag{17}$$

If, in (17), $\frac{h^n}{\lfloor n}\phi^n(x + \theta h)$ denote the sum of all the terms in the right member after the first n, and again $\frac{h^m}{\lfloor m} R$ denote the sum of all the terms after the first m, m being any integer between n and $2n$, we can write the two equations

$$\phi(x+h) = \phi x + h\phi' x + \frac{h^2}{2}\phi'' x + \frac{h^3}{2.3}\phi''' x + \ldots . \frac{h^{n-1}}{\lfloor n-1}\phi^{n-1} x + \frac{h^n}{\lfloor n}\phi^n(x+\theta h). \tag{18}$$

$$\phi(x+h) = \phi x + h\phi' x + \frac{h^2}{2}\phi'' x + \frac{h^3}{2.3}\phi''' x + \ldots . \frac{h^n}{\lfloor n}\phi^n x + \frac{h^{n+1}}{\lfloor n+1}\phi^{n+1} x$$
$$+ \ldots . \frac{h^{m-1}}{\lfloor m-1}\phi^{m-1} x + \frac{h^m}{\lfloor m} R. \tag{19}$$

From (18) and (19),

$$\phi^n(x+\theta h) = \phi^n x + \frac{h}{(n+1)}\phi^{n+1} x + \frac{h^2}{(n+1)(n+2)}\phi^{n+2} x + \ldots . \frac{h^{m-n}}{(n+1)(n+2) \ldots . m} R. \tag{20}$$

By differentiating (18) n times, regarding x as constant,

$$\phi^n(x+h) = \phi^n(x+\theta h) + nh\frac{d\phi^n(x+\theta h)}{dh} + \frac{n(n-1)h^2}{2}\frac{d^2\phi^n(x+\theta h)}{dh^2}$$
$$+ \ldots . \frac{h^n}{\lfloor n}\frac{d^n\phi^n(x+\theta h)}{dh^n}. \tag{21}$$

From (18), when $n=0$,

$$\phi(x+h)=\frac{h^0}{1}\phi^0(x+\theta h)=\phi^0(x+\theta h), \quad \text{and} \quad \theta=1. \tag{22}$$

If $n=\infty$, then, by (20),

$$\phi^n(x+\theta h)=\phi^n x, \text{ and } \theta=0. \tag{23}$$

From (20) and (21) we find the quantity $\phi^n(x+\theta h)$ can never equal either $\phi^n(x+h)$, or $\phi^n x$, unless n has such a value as to cause all the terms after the first, in the right members of (20) and (21), to vanish.

This can only happen in (20) when n is infinite, and in (21) when n is zero. Hence for finite integral values of n, θ is a positive proper fraction, or θ is between zero and a unit.

Differentiating (20), regarding x as constant,

$$\theta\phi^{n+1}(x+\theta h)+\frac{d\theta}{dh}h\phi^{n+1}(x+\theta h)=\frac{1}{n+1}\phi^{n+1}x+\frac{2h}{(n+1)(n+2)}\phi^{n+2}x$$
$$+\frac{3h^2}{(n+1)(n+2)(n+3)}\phi^{n+3}x+\ldots\frac{(m-n)h^{m-n-1}}{(n+1)\ldots m}R'+\frac{h^{m-n}}{(n+1)(n+2)\ldots m}\frac{dR'}{dh}. \tag{24}$$

In (24), when $h=0$,

$$(\theta)_{h=0}\phi^{n+1}x=\frac{1}{n+1}\phi^{n+1}x, \quad \text{and} \quad (\theta)_{h=0}=\frac{1}{n+1}. \tag{25}$$

Continuing the differentiation of (24), regarding x as constant, and in the results putting $h=0$,

$$2\left(\frac{d\theta}{dh}\right)_{h=0}\phi^{n+1}x+(\theta^2)_{h=0}\phi^{n+2}x=\frac{2\phi^{n+2}x}{(n+1)(n+2)}. \tag{26}$$

$$3\left(\frac{d^2\theta}{dh^2}\right)_{h=0}\phi^{n+1}x+6\left(\theta\frac{d\theta}{dh}\right)_{h=0}\phi^{n+2}x+(\theta^3)_{h=0}\phi^{n+3}x=\frac{2.3\,\phi^{n+3}x}{(n+1)(n+2)(n+3)}. \tag{27}$$

$$4\left(\frac{d^3\theta}{dh^3}\right)_{h=0}\phi^{n+1}x+12\left(\theta\frac{d^2\theta}{dh^2}\right)_{h=0}\phi^{n+2}x+12\left(\frac{d\theta}{dh}\right)^2_{h=0}\phi^{n+2}x+12\left(\theta^2\frac{d\theta}{dh}\right)_{h=0}\phi^{n+3}x$$
$$+(\theta^4)_{h=0}\phi^{n+4}x=\frac{2.3.4\,\phi^{n+4}x}{(n+1)(n+2)\ldots(n+4)}. \tag{28}$$

. . . .

From (25) and (26),

$$\left(\frac{d\theta}{dh}\right)_{h=0}=\left(\frac{2(n+1)-(n+2)}{2(n+1)^2.(n+2)}\right)\left(\frac{\phi^{n+2}x}{\phi^{n+1}x}\right). \tag{29}$$

Again, from (27), by the aid of (25) and (29),

$$\left(\frac{d^2\theta}{dh^2}\right)_{h=0}=\left(\frac{2.3(n+1)^2-(n+2)(n+3)}{3(n+1)^3.(n+2)(n+3)}\right)\left(\frac{\phi^{n+3}x}{\phi^{n+1}x}\right)-\left(\frac{2(n+1)-(n+2)}{(n+1)^3.(n+2)}\right)\left(\frac{\phi^{n+2}x}{\phi^{n+1}x}\right)^2. \tag{30}$$

In like manner it may be shown, from (28), that

$$\left(\frac{d^3\theta}{dh^3}\right) = \left(\frac{2.3.4(n+1)^3-(n+2)(n+3)(n+4)}{2^2(n+1)^4.(n+2)(n+3)(n+4)}\right)\left(\frac{\phi^{n+4}x}{\phi^{n+1}x}\right) - \left(\frac{2.3(n+1)-3(n+2)}{2(n+1)^4.(n+2)}\right.$$
$$+\left.\frac{2.3^2(n+1)^2-3(n+2)(n+3)}{3(n+1)^4.(n+2)(n+3)}\right)\left(\frac{\phi^{n+3}x}{\phi^{n+1}x}.\frac{\phi^{n+2}x}{\phi^{n+1}x}\right) - \left(\frac{2^2.3(n+1)^2-2^2.3(n+1)(n+2)+3(n+2)^2}{2^2(n+1)^4.(n+2)^2}\right.$$
$$\left.-\frac{2.3(n+1)-3(n+2)}{(n+1)^4.(n+2)}\right)\left(\frac{\phi^{n+2}x}{\phi^{n+1}x}\right)^2. \quad (31)$$

In the same way, the values of $\left(\frac{d^4\theta}{dh^4}\right)_{h=0}$, $\left(\frac{d^5\theta}{dh^5}\right)_{h=0}$ .... may be found.

From (25), $\theta = \frac{1}{n+1} + \phi$, where $\phi = 0$ when $h = 0$; but by one differentiation with respect to h, and in the result putting $h = 0$,

$$\left(\frac{d\theta}{dh}\right)_{h=0} = \left(\frac{d\phi}{dh}\right)_{h=0} = \left(\frac{2(n+1)-(n+2)}{2(n+1)^2.(n+2)}\right)\left(\frac{\phi^{n+2}x}{\phi^{n+1}x}\right),$$

by (29). Consequently, by one differentiation with respect to h, h has been eliminated from one term in ϕ, and also from a second term in θ. Hence, before differentiation,

$$\theta = \frac{1}{n+1} + h\left(\frac{d\theta}{dh}\right)_{h=0} + \phi' = \frac{1}{n+1} + h\left[\frac{2(n+1)-(n+2)}{2(n+1)^2.(n+2)}\left(\frac{\phi^{n+2}x}{\phi^{n+1}x}\right)\right] + \phi',$$

where ϕ' and $\left(\frac{d\phi'}{dh}\right)$ both vanish when $h = 0$. Differentiating the last equation twice, regarding x as constant, and in the result putting $h = 0$, $\left(\frac{d^2\theta_1}{dh^2}\right)_{h=0}$ $= \left(\frac{d^2\phi'}{dh^2}\right)_{h=0} =$ the right member (30). Hence by two differentiations with respect to h; $\frac{h^2}{2}$ has been eliminated from a term in ϕ', and from a third term in θ. Consequently,

$$\theta = \frac{1}{n+1} + h\left(\frac{d\theta}{dh}\right)_{h=0} + \frac{h^2}{2}\left(\frac{d^2\theta}{dh^2}\right)_{h=0} + \phi'' = \frac{1}{n+1} + h\left[\left(\frac{2(n+1)-(n+2)}{2(n+1)^2.(n+2)}\right)\left(\frac{\phi^{n+2}x}{\phi^{n+1}x}\right)\right]$$
$$+\frac{h^2}{2}\left[\left(\frac{2.3(n+1)^2-(n+2)(n+3)}{3(n+1)^3.(n+2).(n+3)}\right)\left(\frac{\phi^{n+3}x}{\phi^{n+1}x}\right) - \left(\frac{2(n+1)-(n+2)}{(n+1)^3.(n+2)}\right)\left(\frac{\phi^{n+2}x}{\phi^{n+1}x}\right)^2\right] + \phi'',$$

where ϕ'', $\frac{d\phi''}{dh}$, and $\frac{d^2\phi''}{dh^2}$ vanish when $h = 0$. In like manner additional terms may be found, and the general form of θ shown to be,

$$\theta = \frac{1}{n+1} + h\left(\frac{d\theta}{dh}\right)_{h=0} + \frac{h^2}{2}\left(\frac{d^2\theta}{dh^2}\right)_{h=0} + \frac{h^3}{2.3}\left(\frac{d^3\theta}{dh^3}\right)_{h=0} + \dots \frac{h^{n-1}}{\lfloor n-1}\left(\frac{d^{n-1}\theta}{dh^{n-1}}\right)_{h=0} + \dots, \quad (32)$$

where $\left(\frac{d\theta}{dh}\right)_{h=0}$, $\left(\frac{d^2\theta}{dh^2}\right)_{h=0}$, ... are expressible in terms $\phi^{n+1}x$, $\phi^{n+2}x$, ... $\phi^{2n}x$, as in (29), (30), (31).

From (11), (12), ... (16), it may readily be shown that

$$\left(\frac{d\theta_1}{dh}\right)_{h=0} = \frac{1}{24}\frac{\phi'''x}{\phi''x}. \tag{33}$$

$$\left(\frac{d^2\theta_1}{dh^2}\right)_{h=0} = \frac{1}{24}\frac{\phi^{\text{IV}}x}{\phi''x} - \frac{1}{24}\left(\frac{\phi'''x}{\phi''x}\right)^2. \tag{34}$$

$$\left(\frac{d^3\theta_1}{dh^3}\right)_{h=0} = \frac{11}{320}\frac{\phi^{\text{V}}x}{\phi''x} - \frac{3}{32}\frac{\phi^{\text{IV}}x}{\phi''x}\frac{\phi'''x}{\phi''x} + \frac{11}{192}\left(\frac{\phi'''x}{\phi''x}\right)^3. \tag{35}$$

$$\left(\frac{d^4\theta_1}{dh^4}\right)_{h=0} = \frac{13}{480}\frac{\phi^{\text{VI}}x}{\phi''x} - \frac{43}{480}\frac{\phi^{\text{V}}x}{\phi''x}\frac{\phi'''x}{\phi''x} + \frac{7}{32}\frac{\phi^{\text{IV}}x}{\phi''x}\left(\frac{\phi'''x}{\phi''x}\right)^2 - \frac{1}{16}\left(\frac{\phi^{\text{IV}}x}{\phi''x}\right)^2 - \frac{3}{32}\left(\frac{\phi'''x}{\phi''x}\right)^4, \tag{36}$$

$$\left(\frac{d^5\theta_1}{dh^5}\right)_{h=0} = \frac{19}{896}\frac{\phi^{\text{VII}}x}{\phi''x} - \frac{31}{384}\frac{\phi^{\text{VI}}x}{\phi''x}\frac{\phi'''x}{\phi''x} + \frac{15}{64}\frac{\phi^{\text{V}}x}{\phi''x}\left(\frac{\phi'''x}{\phi''x}\right)^2 - \frac{55}{108}\frac{\phi^{\text{IV}}x}{\phi''x}\left(\frac{\phi'''x}{\phi''x}\right)^3 + \frac{5}{16}\left(\frac{\phi^{\text{IV}}x}{\phi''x}\right)^2\frac{\phi'''x}{\phi''x}$$
$$- \frac{53}{384}\frac{\phi^{\text{V}}x}{\phi''x}\frac{\phi^{\text{IV}}x}{\phi''x} + \frac{185}{1152}\left(\frac{\phi'''x}{\phi''x}\right)^5. \tag{37}$$

. . . .

The value of θ_1 may be found from those of

$$\left(\frac{d\theta_1}{dh}\right)_{h=0}, \left(\frac{d^2\theta_1}{dh^2}\right)_{h=0}, \left(\frac{d^3\theta_1}{dh^3}\right)_{h=0}, \ldots$$

in the same manner that the value of θ was derived from those of

$$\left(\frac{d\theta}{dh}\right)_{h=0}, \left(\frac{d^2\theta}{dh^2}\right)_{h=0}, \ldots$$

Hence,

$$\theta_1 = \frac{1}{2} + h\left[\frac{1}{24}\frac{\phi'''x}{\phi''x}\right] + \frac{h^2}{2}\left[\frac{1}{24}\frac{\phi^{\text{IV}}x}{\phi''x} - \frac{1}{24}\left(\frac{\phi'''x}{\phi''x}\right)^2\right] + \ldots, \tag{38}$$

or,

$$\theta_1 = \frac{1}{2} + h\left(\frac{d\theta_1}{dh}\right)_{h=0} + \frac{h^2}{2}\left(\frac{d^2\theta_1}{dh^2}\right)_{h=0} + \frac{h^3}{2.3}\left(\frac{d^3\theta_1}{dh^3}\right)_{h=0} + \ldots \frac{h^n}{\lfloor n}\left(\frac{d^n\theta_1}{dh^n}\right)_{h=0} + \ldots, \tag{39}$$

where

$$\left(\frac{d\theta_1}{dh}\right)_{h=0}, \left(\frac{d^2\theta_1}{dh^2}\right)_{h=0}, \ldots \left(\frac{d^n\theta_1}{dh^n}\right)_{h=0}$$

are expressible in terms of $\phi''x$, $\phi'''x$, ... $\phi^{n+2}x$.

If in (5), (6), (7), (8) and (9) we write θ_1 for m, θ_2 for θ, and advance each order of diff. co. of x to the next higher, we obtain the same results as when

$h=0$ in the successive diff. co. with respect to h, of $(b)'$, and these results by easy reductions give

$$\left(\frac{d\theta_2}{dh}\right)_{h=0}=\frac{1}{96}\frac{\phi^{\text{IV}}x}{\phi'''x}+\frac{1}{48}\frac{\phi'''x}{\phi''x}. \tag{40}$$

$$\left(\frac{d^2\theta_2}{dh^2}\right)_{h=0}=\frac{1}{192}\frac{\phi^{\text{V}}x}{\phi'''x}-\frac{1}{192}\left(\frac{\phi^{\text{IV}}x}{\phi'''x}\right)^2+\frac{7}{288}\frac{\phi^{\text{IV}}x}{\phi''x}-\frac{1}{48}\left(\frac{\phi'''x}{\phi''x}\right)^2. \tag{41}$$

$$\left(\frac{d^3\theta_2}{dh^3}\right)_{h=0}=\frac{11}{5120}\frac{\phi^{\text{VI}}x}{\phi'''x}-\frac{3}{512}\frac{\phi^{\text{V}}x}{\phi'''x}\frac{\phi^{\text{IV}}x}{\phi'''x}+\frac{11}{3072}\left(\frac{\phi^{\text{IV}}x}{\phi'''x}\right)^2+\frac{27}{1280}\frac{\phi^{\text{V}}x}{\phi''x}+\frac{1}{768}\frac{\phi^{\text{IV}}x}{\phi''x}\frac{\phi^{\text{IV}}x}{\phi'''x}$$
$$-\frac{119}{2304}\frac{\phi^{\text{IV}}x}{\phi''x}\frac{\phi'''x}{\phi''x}+\frac{11}{384}\left(\frac{\phi'''x}{\phi''x}\right)^3. \tag{42}$$

. . . .

From these values of $\left(\frac{d\theta_2}{dh}\right)_{h=0}$, $\left(\frac{d^2\theta_2}{dh^2}\right)_{h=0}$, ..., we find

$$\theta_2=\frac{1}{4}+h\left[\frac{1}{96}\frac{\phi^{\text{IV}}x}{\phi'''x}+\frac{1}{48}\frac{\phi'''x}{\phi''x}\right]+\frac{h^2}{2}\left[\frac{1}{192}\frac{\phi^{\text{V}}x}{\phi'''x}-\frac{1}{192}\left(\frac{\phi^{\text{IV}}x}{\phi'''x}\right)^2+\frac{7}{288}\frac{\phi^{\text{IV}}x}{\phi''x}-\frac{1}{48}\left(\frac{\phi'''x}{\phi''x}\right)^2\right]$$
$$+\ldots. \tag{43}$$

or,

$$\theta_2=\frac{1}{4}+h\left(\frac{d\theta_2}{dh}\right)_{h=0}+\frac{h^2}{2}\left(\frac{d^2\theta_2}{dh^2}\right)_{h=0}+\frac{h^3}{2.3}\left(\frac{d^3\theta_2}{dh^3}\right)_{h=0}+\ldots.\frac{h^n}{\underline{|n}}\left(\frac{d^n\theta_2}{dh^n}\right)_{h=0}+\ldots., \tag{44}$$

where

$$\left(\frac{d\theta_2}{dh}\right)_{h=0},\ \left(\frac{d^2\theta_2}{dh^2}\right)_{h=0},\ \ldots.\ \left(\frac{d^n\theta_2}{dh^n}\right)_{h=0}$$

are expressible in terms of $\phi''x$, $\phi'''x$, ... $\phi^{n+3}x$, as in (40), (41), (42).

If in (5), (6), (7), (8) and (9) we write θ_2 for m, θ_3 for θ, $\phi'''x$ for $\phi'x$, $\phi^{\text{IV}}x$ for $\phi''x$, and so on, then reduce the results, we get

$$\left(\frac{d\theta_3}{dh}\right)_{h=0}=\frac{1}{384}\frac{\phi^{\text{V}}x}{\phi^{\text{IV}}x}+\frac{1}{192}\frac{\phi^{\text{IV}}x}{\phi'''x}+\frac{1}{96}\frac{\phi'''x}{\phi''x}. \tag{45}$$

$$\left(\frac{d^2\theta_3}{dh^2}\right)_{h=0}=\frac{1}{1536}\frac{\phi^{\text{VI}}x}{\phi^{\text{IV}}x}-\frac{1}{1536}\left(\frac{\phi^{\text{V}}x}{\phi^{\text{IV}}x}\right)^2+\frac{7}{2304}\frac{\phi^{\text{V}}x}{\phi'''x}-\frac{1}{384}\left(\frac{\phi^{\text{IV}}x}{\phi'''x}\right)^2+\frac{7}{576}\frac{\phi^{\text{IV}}x}{\phi''x}+\frac{1}{1152}\frac{\phi^{\text{V}}x}{\phi^{\text{IV}}x}\frac{\phi'''x}{\phi''x}$$
$$-\frac{1}{96}\left(\frac{\phi'''x}{\phi''x}\right)^2. \tag{46}$$

. . . .

From these values of $\left(\frac{d\theta_3}{dh}\right)_{h=0}$, $\left(\frac{d^2\theta_3}{dh^2}\right)_{h=0}$, ..., we find

$$\theta_3=\frac{1}{8}+h\left(\frac{d\theta_3}{dh}\right)_{h=0}+\frac{h^2}{2}\left(\frac{d^2\theta_3}{dh^2}\right)_{h=0}+\ldots.\frac{h^n}{\underline{|n}}\left(\frac{d^n\theta_3}{dh^n}\right)_{h=0}+\ldots. \tag{47}$$

where $\left(\frac{d\theta_3}{dh}\right)_{h=0}$, $\left(\frac{d^2\theta_3}{dh^2}\right)_{h=0}$, ... are expressible in terms of $\phi''x$, $\phi'''x$, ... $\phi^{n+4}x$, as in (45), (46).

In general, it may be shown that

$$\theta_n = \frac{1}{2^n} + h\left(\frac{d\theta_n}{dh}\right)_{h=0} + \frac{h^2}{2}\left(\frac{d^2\theta_n}{dh^2}\right)_{h=0} + \frac{h^3}{2.3}\left(\frac{d^3\theta_n}{dh^3}\right)_{h=0} + \dots \frac{h^{n-1}}{\lfloor n-1}\left(\frac{d^{n-1}\theta_n}{dh^{n-1}}\right)_{h=0} + \dots, \quad (48)$$

where $\left(\frac{d\theta_n}{dh}\right)_{h=0}$, $\left(\frac{d^2\theta_n}{dh^2}\right)_{h=0}$, ... $\left(\frac{d^{n-1}\theta_n}{dh^{n-1}}\right)_{h=0}$ can be expressed in terms of $\phi''x$, $\phi'''x$, ... $\phi^{2n}x$, as in preceding cases.

Eliminating $\phi'(x+\theta_1 h)$, $\phi''(x+\theta_2 h)$, ... $\phi^{n-1}(x+\theta_{n-1}h)$ from $(a)'$, $(b)'$, ..., there results

$$\phi(x+h) = \phi x + h\phi' x + \theta_1 h^2 \phi'' x + \theta_1\theta_2 h^3 \phi''' x + \dots \theta_1\theta_2\theta_3 \dots \theta_{n-1} h^n \phi^n(x+\theta_n h). \quad (49)$$

Since θ_1 contains h, h^2, h^3, ... as factors, and does not contain h in any other form, and since the same holds true in the expressions giving the values of θ_2, θ_3, ..., it is plain that (49) will contain h as a factor in the regular ascending integral powers, h, h^2, h^3, ..., and will not contain h in any other form, when θ_1, θ_2, ... are replaced by their values in terms of x and h.

From (39), (44), (47), and (48), we readily find

$$\theta_1 h^2 \phi'' x = h^2\phi'' x\left[\frac{1}{2} + h\left(\frac{d\theta_1}{dh}\right)_{h=0} + \frac{h^2}{2}\left(\frac{d^2\theta_1}{dh^2}\right)_{h=0} + \frac{h^3}{2.3}\left(\frac{d^3\theta_1}{dh^3}\right)_{h=0}\right] + \text{terms containing } h^6, h^7, \dots \quad (50)$$

$$\theta_1\theta_2 h^3 \phi''' x = h^3\phi''' x\left[\frac{1}{8} + \frac{h}{4}\left(\frac{d\theta_1}{dh}\frac{2\,d\theta_2}{dh}\right)_{h=0} + \frac{h^2}{8}\left(\frac{d^2\theta_1}{dh^2} + \frac{8\,d\theta_1 d\theta_2}{dh\,dh} + \frac{2\,d^2\theta_2}{dh^2}\right)_{h=0}\right] + \text{terms containing } h^6, h^7, \dots \quad (51)$$

$$\theta_1\theta_2\theta_3 h^4 \phi^{\text{IV}} x = h^4\phi^{\text{IV}} x\left[\frac{1}{16} + \frac{h}{32}\left(\frac{d\theta_1}{dh} + \frac{2\,d\theta_2}{dh} + \frac{4\,d\theta_3}{dh}\right)_{h=0}\right] + \text{terms containing } h^6, h^7, \dots \quad (52)$$

$$\theta_1\theta_2\theta_3\theta_4 h^5 \phi^{\text{V}} x = h^5\phi^{\text{V}} x\left[\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{8}\cdot\frac{1}{16}\right] + \text{terms containing } h^6, h^7, \dots \quad (53)$$

In (50), (51), (52), (53), replacing the several diff. co. of θ_1, θ_2, ... by their values in terms of x and h, and substituting the results for $\theta_1 h^2\phi'' x$, $\theta_1\theta_2 h^3 \phi''' x$, ... in (49), we get

$$
\begin{aligned}
\phi(x+h) = \phi x + h\phi' x + \frac{h^2}{2}\phi'' x + \frac{h^3}{24}\phi''' x + \frac{h^4}{48}\phi^{\mathrm{IV}} x - \frac{h^4}{48}\frac{(\phi''' x)^2}{\phi'' x} + \frac{11\,h^5}{1920}\phi^{\mathrm{V}} x - \frac{3\,h^5}{192}\frac{\phi''' x}{\phi'' x} + \frac{11\,h^5}{1152}\frac{(\phi''' x)^3}{(\phi'' x)^2} \\
+ \frac{h^3}{8}\phi''' x + \frac{h^4}{192}\phi^{\mathrm{IV}} x + \frac{h^4}{96}\frac{(\phi''' x)^2}{\phi'' x} + \frac{h^5}{768}\phi^{\mathrm{V}} x + \frac{h^5}{192}\frac{\phi''' x}{\phi'' x} - \frac{h^5}{192}\frac{(\phi''' x)^3}{(\phi'' x)^2} \\
+ \frac{h^4}{64}\phi^{\mathrm{IV}} x + \frac{h^4}{96}\frac{(\phi''' x)^2}{\phi'' x} + \frac{h^5}{3072}\phi^{\mathrm{V}} x + \frac{h^5}{2304}\frac{\phi''' x}{\phi'' x} + \frac{h^5}{1152}\frac{(\phi''' x)^3}{(\phi'' x)^2} \\
+ \frac{h^5}{1024}\phi^{\mathrm{V}} x + \frac{7\,h^5}{1152}\frac{\phi''' x}{\phi'' x} - \frac{h^5}{192}\frac{(\phi''' x)^3}{(\phi'' x)^2} \\
+ \frac{h^5}{768}\frac{\phi''' x}{\phi'' x} \\
+ \frac{h^5}{768}\frac{\phi''' x}{\phi'' x} \\
+ \frac{h^5}{768}\frac{\phi''' x}{\phi'' x} \\
- \frac{h^5}{768}\frac{(\phi^{\mathrm{IV}} x)^2}{\phi'' x} + \text{terms containing } h^6, h^7, \ldots, \\
+ \frac{h^5}{1536}\frac{(\phi^{\mathrm{IV}} x)^2}{\phi'' x} \\
+ \frac{h^5}{1536}\frac{(\phi^{\mathrm{IV}} x)^2}{\phi'' x}.
\end{aligned}
$$

Uniting terms,

$$
\phi(x+h) = \phi x + h\phi' x + \frac{h^2}{2}\phi'' x + \frac{h^3}{2.3}\phi''' x + \frac{h^4}{2.3.4}\phi^{\mathrm{IV}} x + \frac{h^5}{2.3.4.5}\phi^{\mathrm{V}} x + \text{terms containing } h^6, h^7, \ldots.
$$

By extending the work, it may be shown that $\phi(x+h) = \phi x + h\phi' x + \frac{h^2}{2}\phi'' x + \frac{h^3}{2.3}\phi''' x + \ldots \frac{h^n}{\lfloor n}\phi^n x +$ such terms in $\theta_1 h^2 \phi'' x$, $\theta_1\theta_2 h^3 \phi''' x, \ldots \theta_1\theta_2, \ldots \theta_{n-1} h^n \phi^n(x+\theta_n h)$ as contain $h^{n+1}, h^{n+2}, \ldots$, but no power of h less than h^{n+1}. The sum of all the terms in the right member of the last equation, after the first n, may be denoted by $\frac{h^n}{\lfloor n}\phi^n(x+\theta h)$, giving (18), and the value of θ found as before.

On the Theory of Rational Derivation on a Cubic Curve.

By William E. Story.

Theory of Indices.

In a recent number of this journal* Professor Sylvester has given the elements of a theory of *rational derivation* on a cubic curve, i. e. a theory of those points of the curve whose co-ordinates can be expressed as rational functions of an arbitrary *initial* point of the same; a theory which, although devised for the purpose of solving an arithmetical problem,† has an interest of its own from a geometrical point of view. It is, so far as it goes, the essence of the theory of the representation of the points on a cubic by means of a single parameter, i. e. its methods are substantially the same as those which have been employed in the development of that theory, but it does not assume any such actual representation. Previous to the above-mentioned foundation of this theory no one had ever, so far as I know, considered other rational derivatives of a point than its tangentials of various orders.

In this paper I propose to develop this new *theory of indices* in a more general and symmetrical form than that originally given to it; and, finally, by combining it with the theory of parameters, I shall solve a number of problems especially relating to the enumeration of points having certain properties analogous to those of singular points or of the contacts of singular tangents.

I shall call, with Professor Sylvester, the point in which the junction of two points of the cubic again meets the curve the *connective* of those two points; then it is evident that the connective of any two rational derivatives of a common initial point is also a rational derivative of the same initial. The tangential of a rational derivative is only a special case of such a connective, viz. the connective of the rational derivative with itself. It is by this method of collineation that Professor Sylvester obtains the derivatives. That such a method will give *all* the rational derivatives of a point is not yet proved, and the question is irrelevant to

* This volume, pages 58 – 88.

† Namely, from one solution in integers of the equation $x^3 + y^3 + z^3 + kxyz = 0$, to find others.

the present investigation; but, in view of the results obtained, it is difficult to see how any geometrical method can give other rational derivatives; at all events, I shall for the present use the expression *derivative* to designate such a derivative only. The derivatives which are thus obtained are the points to which correspond, as I shall show, values of the parameter differing only by multiples of certain periods (the inflexion-periods) from commensurable values, if the parameter μ be so chosen, as it always may be, that $\mu + \mu' + \mu'' = 0$ is the condition for three collinear points. The difference in method of the theories of *indices* and *parameters* consists in this: that in the latter continuous values of a parameter are assigned to the continuous points of the curve in accordance with its equation, while in the former to an arbitrary point taken as the initial an index 1 is assigned, and then to its derivatives in a certain order all positive and negative integers as indices. The index of a derivative thus expresses (with a certain modification due to the inflexion-periods) the number by which the parameter of the initial must be multiplied in order to obtain the parameter of the derivative. Viewed from an algebraic standpoint, as Professor Sylvester has shown, the square of the index of a derivative on a non-singular cubic is the degree of the co-ordinates of the derivative in terms of the initial; and from a geometrical standpoint, as is proved in the sequel, it is the number of points which bear to any given point the relation of initial to derivative with the index in question. This method of indices is particularly useful in determining the *number* of points whose derivatives satisfy certain conditions, and for such a determination it is in general necessary to take into account the periodicity of the parametric representation; at least it is necessary to distinguish the cases in which there is no period, or one or two periods. The advantage of considering the periodicity need not be lost in using the method of indices, so long as the problem in hand does not involve the actual determination of points, but only their number. If the question of reality enters, of course the nature of the periods and their relations to the co-ordinates must be considered. The condition of collineation above cited must be our guide in the assignment of the indices, in order that the relation mentioned may subsist between index and parameter,* i. e. for the indices a, b, c of three collinear points the fundamental formula holds,

$$a + b + c = 0,$$

* For the condition of collineation might have been chosen any linear relation between $a + b + c$, $bc + ca + ab$, abc and an arbitrary constant, i. e. any linear relation between the coefficients of the cubic equation of which a, b, c may be regarded as the roots; but it is more convenient to consider this condition as put, by a proper transformation, which is not in general algebraic, into the form given above. The indices of all the rational derivatives of any point will then be integers; otherwise they will be commensurable fractions.

or
$$[a, b] = -(a + b), \tag{1}$$
if, in general, $[a, b]$ denote the index of the connective of two points whose indices are a and b. In particular, for the tangential of a point whose index is a we have
$$[a, a] = -2a, \tag{2}$$
and for an inflexion
$$-2a = a, \text{ i.e. } a = 0, \tag{3}$$
so that each inflexion has the index 0. I shall have occasion hereafter to distinguish derivatives having a common index by means of suffices, but at present we will confine our attention to one point corresponding to each index, and to one inflexion selected at pleasure (a real inflexion, if we are to consider real derivatives) and give it the index 0, so that it will not yet be necessary to use any distinguishing mark. The connective of any point with an inflexion may be called the *opposite* of that point with respect to the inflexion. For the opposite with respect to 0 of a point whose index is a we have
$$[a, 0] = -a. \tag{4}$$
We know that the opposites of three collinear points with respect to the same inflexion are also collinear; hence, if $[a, b] = -(a + b)$ for any particular values of a and b, then
$$[-a, -b] = (a + b). \tag{5}$$

It is necessary to show that the indices can be assigned so that the fundamental formula (1) shall hold for any two derivatives and their connective. The application of (4) and (5) will make it unnecessary to prove the formula separately for all cases; viz., every number of the form $3m - 1$, in which m is any positive or negative integer, is the negative of a number of the form $3m + 1$; after I have assigned all the indices of the form $3m + 1$, I shall assign those of the form $3m - 1$ by (4); and when the fundamental formula has been proved for the connective of any two indices of the form $3m + 1$, it is shown by (5) to hold for two indices of the form $3m - 1$. It will be noticed that, if a and b are both of one of the forms $3m + 1$, $3m - 1$, $3m$, then $-(a + b)$ is also of that form; and if a and b are of different forms, $-(a + b)$ is of the form different from either.

The proof of the fundamental formula, after the indices have been assigned, depends upon a special form of a theorem of Professor Sylvester's (Salmon's Higher Plane Curves, 2d ed., page 135), which may be put thus: *If four points on a cubic be grouped in pairs in any way, the connective of the connectives of the points in the separate pairs is independent of the manner of grouping.* The point which thus

depends only on the position of the four given points is their *coresidual*, and its index may be denoted by $[a, b, c, d]$, where a, b, c, d are the indices of the four points. The theorem just stated may then be expressed thus:—

$$\big[[a, b], [c, d]\big] = \big[[a, c], [b, d]\big] = \big[[a, d], [b, c]\big] = [a, b, c, d]. \qquad (6)$$

For convenience I omit the inside brackets and write the indices in the order in which they are used in pairs, separating them by commas; thus (6) becomes

$$[a, b, c, d] = [a, c, b, d] = [a, d, b, c].$$

Later I shall speak also of the coresidual of $3n + 1$ indices instead of the index of the coresidual of $3n + 1$ points. The indices of the form $3n + 1$ are assigned thus:—

$$1 = \text{index of the initial},$$

$$[1, 1] = -2 = \text{index of the tangential of the initial},$$

and then, by the use of -2 and 1 alternately,

$$\begin{array}{ll} [-2, -2] = 4, & [\ 4, 1] = -5, \\ [-5, -2] = 7, & [\ 7, 1] = -8, \\ [-8, -2] = 10, & [10, 1] = -11, \text{ etc.,} \end{array}$$

from which follow

$$\begin{array}{ll} & [-\ 2, 1] = 1, \\ [\ 4, -2] = -2, & [-\ 5, 1] = 4, \\ [\ 7, -2] = -5, & [-\ 8, 1] = 7, \\ [10, -2] = -8, & [-11, 1] = 10, \text{ etc.,} \end{array}$$

so that, if a is any index of the form $3m + 1$,

$$[a, 1] = -(a + 1) \quad \text{and} \quad [a, -2] = -(a - 2). \qquad (7)$$

If a and b are any two indices of the form $3m + 1$, then by (7) and (6)

$$\begin{aligned} [a, b] &= [-(a + 1), 1 - (b - 2), -2] = [-(a + 1), -2, -(b - 2), 1] \\ &= [a + 3, b - 3], \end{aligned} \qquad (8)$$

so that the index of the connective* of two indices remains unchanged if either index be increased by any multiple of 3, while the other is decreased by the

* It is a convenient abbreviation to speak of the connective of two *indices* instead of that of the corresponding points.

same multiple of 3, the sum remaining constant. Whatever value $\equiv 1 \pmod{3}$ either index may have, it can be increased if negative, or decreased if positive, by such a multiple of 3 that it shall become 1, so that (a and b both of the form $3m+1$)

$$[a, b] = [a+b-1, 1] = -(a+b);$$

and hence by (5)

$$[-a, -b] = (a+b),$$

i.e. the fundamental formula holds for any two indices, both of the form $3m+1$, or both of the form $3m-1$. I will introduce multiples of 3 by the formula

$$[3m-1, 1] = -3m, \tag{9}$$

for all positive and negative values of m, in accordance with the fundamental theorem. Then also, for all values of m,

$$[3m, 1] = -(3m+1). \tag{10}$$

Then by (1), (4), (6), and (9), of which (1) and (4) are now known to be true for indices of the form $3m+1$ and of the form $3m-1$,

$$[3m+1, -1] = [-3m+1, -2, 1, 0] = [-3m+1, 0, -2, 1] = [3m-1, 1] \\ = -3m, \tag{11}$$

from which follows

$$[3m, -1] = -(3m-1). \tag{12}$$

Moreover

$$[3m, 0] = [-3m+1, -1, 0, 0] = [-3m+1, 0, -1, 0] = [3m-1, 1] \\ = -3m. \tag{13}$$

It has now been proved that, for *all* values of a,

$$[a, 1] = -(a+1), \quad [a, -1] = -(a-1), \quad \text{and} \quad [a, 0] = -a; \tag{14}$$

whence, for *all* values of a and b,

$$[a,b]=[-a-1,1,-b+1,-1]=[-a-1,-1,-b+1,1]=[a+2,b-2], \tag{15}$$

which can be repeated any number of times until one or the other of the two indices becomes 0 or 1, in either of which cases the fundamental formula holds; hence it holds for all values of a and b, i.e.

I. *The connective of any two indices is their negative sum.*

For the coresidual of four indices a, b, c, d we have

$$[a, b, c, d] = [-a-b, -c-d] = a+b+c+d, \tag{16}$$

i. e.

II. *The coresidual of four indices is their sum.*

In general (see Salmon's Higher Plane Curves, 2d ed., §§ 154–161), given any $3n-1$ points on a cubic, all the curves of the order n which can be passed through them will intersect the cubic again in a fixed point, the *residual* of the given $3n-1$ points; and given any $3n+1$ points on a cubic, any curve of the order $n+1$ which can be passed through them will intersect the cubic again in two points whose connective is a fixed point, the *coresidual* of the given $3n+1$ points. Furthermore it is substantially proved (Salmon, l. c.) that, in the determination of residual or coresidual, for any group of $3k+1$ points can be substituted their coresidual; hence, in the determination of the index of the residual or coresidual of any number of points of the numbered scale, for any four indices we may substitute their sum. We may then group the given indices together by fours as far as possible, and substitute for each four their sum, thus reducing the number of indices by a multiple of 3, without altering their sum. The same reduction may be made in the system of indices thus obtained, and this process carried on until the number of indices is less than four. We speak only of a *residual* of $3n-1$ points and the *coresidual* of $3n+1$ points; so that, after the above reduction in the number of points by a multiple of 3, there will result two points whose residual is their negative sum, or a single point which is the coresidual of the given $3n+1$ points. Hence

III. *The residual of any* $3n-1$ *indices is their negative sum, and the coresidual of any* $3n+1$ *indices is their sum.*

It may be added that the sum of any $3n-1$ indices is the coresidual of the group formed by annexing to them the inflexion 0 twice; the negative sum of any $3n+1$ indices is the residual of the group formed by annexing to them the inflexion 0; the sum of any $3n$ indices is the coresidual of the group formed by annexing to them the inflexion 0; and the negative sum of $3n$ indices is the residual of the group formed by annexing to them the inflexion 0 twice, i. e. is the $3(n+1)^{\text{th}}$ intersection with the cubic of every curve of the order $n+1$ which passes through the $3n$ points, and is tangent to the cubic at the inflexion 0.

Theorem II. can be put into this simple form: *The sum of the indices of the* $3n$ *points of intersection of any* n*-thic with the cubic is* 0.*

* Compare Clebsch, *Vorlesungen über Geometrie*, edited by Lindemann, Vol. I. p. 623.

The foregoing is the complete theory for cuspidal cubics, which have only one point of inflexion; but on a cubic having more than one inflexion, a series of derivatives whose indices are of the form $3m$ and $3m-1$ exists for each inflexion, and these series determine by collineation yet other series whose indices are of the form $3m+1$.

An acnodal or crunodal cubic will have three collinear inflexions; let their indices be $0_0, 0_1, 0_2$ (the numerical value of the index of any inflexion is 0, as above proved); let the indices a_0, for all integral values of a, be assigned as in the preceding case the indices a were, i. e. let 1_0 be the index of any point on the curve, let $-2_0, 4_0, -5_0, 7_0, \ldots$ be assigned with respect to 1_0 as $-2, 4, -5, 7, \ldots$ were assigned with respect to 1, and let $-1_0, 2_0, -4_0, 5_0, \ldots$ be assigned with respect to 1_0 and 0_0 as above $-1, 2, 3, -3, -4, 8, \ldots$ were with respect to 1 and 0. Further, let

$$[a_0, 0_2] = -a_1 \quad \text{and} \quad [a_0, 0_1] = -a_2 \tag{17}$$

serve for the assignment of all indices with the suffices 1 and 2. Then, for all values of a and b,

$$[0_0, 0_0] = 0_0, [0_1, 0_1] = 0_1, [0_2, 0_2] = 0_2,$$
$$[0_1, 0_2] = 0_0, [0_2, 0_0] = 0_1, [0_0, 0_1] = 0_2,$$
$$[a_0, b_0] = -(a+b)_0,$$
$$[a_1, b_2] = [-a_0, 0_2, -b_0, 0_1] = [-a_0, -b_0, 0_1, 0_2] = [(a+b)_0, 0_0] = -(a+b)_0,$$
$$[a_2, b_0] = [-a_0, 0_1, -b_0, 0_0] = [-a_0, -b_0, 0_0, 0_1] = [(a+b)_0, 0_2] = -(a+b)_1,$$
$$[a_0, b_1] = [-a_0, 0_0, -b_0, 0_2] = [-a_0, -b_0, 0_2, 0_0] = [(a+b)_0, 0_1] = -(a+b)_2,$$
$$[a_1, b_1] = [-a_0, 0_2, -b_0, 0_2] = [-a_0, -b_0, 0_2, 0_2] = [(a+b)_0, 0_2] = -(a+b)_1,$$
$$[a_2, b_2] = [-a_0, 0_1, -b_0, 0_1] = [-a_0, -b_0, 0_1, 0_1] = [(a+b)_0, 0_1] = -(a+b)_2;$$

so that, in general, if $\rho(x)$ denotes the minimum positive residue of x (mod. 3), we shall have

$$[a_p, b_q] = -(a+b)_{\rho[-(p+q)]}, \tag{18}$$

where each of the numbers p and q has one of the values 0, 1, 2. Evidently then

$$\begin{aligned}[a_p, b_q, c_r, d_s] &= [-(a+b)_{\rho[-(p+q)]}, -(c+d)_{\rho[-(r+s)]}] \\ &= [a+b+c+d]_{\rho(p+q+r+s)}.\end{aligned} \tag{19}$$

By a simple extension of these formulæ, as in the case of the cuspidal cubic, to the residual or coresidual of any number of points, we have

IV. *The residual of any* $3\text{n} - 1$ *indices on a crunodal or acnodal cubic is their negative sum with a suffix equal to the minimum positive residue* (*mod.* 3) *of the negative sum of their suffices; and the coresidual of any* $3\text{n} + 1$ *indices on such a cubic is the sum of their indices with a suffix equal to the minimum positive residue* (*mod.* 3) *of the sum of their suffices.*

The presence of an inflexion in a group of derivatives can only affect the number of indices in the group and the suffix of the resultant index, i. e. can only have an effect in determining whether the result is residual or coresidual, and hence whether the sum of the indices and the sum of the suffices are to be taken with the positive or negative sign. With this in mind it is easy to make additions to the theorem just given analogous to those which we made to Theorem II., respecting the meaning of the positive sum of $3n - 1$ indices, the negative sum of $3n + 1$ indices, and the positive and negative sums of $3n$ indices, each with either possible suffix. For instance, the sum of $3n$ indices with a suffix equal to the minimum positive residue (mod. 3) of the sum of their suffices is the coresidual of the group formed by annexing to the given $3n$ points the inflexion 0_0; the same sum with a suffix one greater (mod. 3) is the coresidual of the group formed by annexing to the $3n$ points the inflexion 0_1; the negative sum of $3n$ indices with a suffix equal to the minimum positive residue (mod. 3) of the negative of the sum of their suffices is the $3(n+1)^{\text{th}}$ point of intersection with the cubic of every curve of the order $n+1$, which passes through the $3n$ given points and touches the cubic at the point of inflexion 0_0, and also of every curve of the same order which passes through the $3n$ given points and through the inflexions 0_1 and 0_2; the same index with the suffix increased by 1 will be that of the point of intersection when 0_1 is the point of contact or the curve passes through 0_0 and 0_2, etc.

These theorems are true for the derivatives obtained by means of any three collinear inflexions, if three or more exist, and will therefore be true for non-singular cubics as well as for acnodal and crunodal cubics. Non-singular cubics have nine points of inflexion lying by threes in twelve straight lines; designating any three inflexions which are not collinear by 0_{00}, 0_{01}, 0_{10}, in accordance with the above principles I will assign double indices to each of the other six inflexions (using only 0, 1, 2 as suffices) so that the conditions of collineation for $0_{p,q}$, $0_{r,s}$, $0_{t,u}$ shall be $p + r + t = 0$ and $q + s + u = 0$, or, using $\rho(x)$ as above,

$$[0_{p,q}, 0_{r,s}] = 0_{\rho[\ (p+r)],\rho[-(q+s)]}.$$

The twelve sets of collinear [illegible] are then

[illegible]

[illegible] (24)

[illegible]

namely [illegible] are [illegible] successively by the first six combinations and the other six [illegible] —

$[\ldots] = [\ldots] = [\ldots] = [\ldots] = i_1$ [illegible]

[illegible]

What I have represented in the preceding case by [illegible] I will represent in the case of the non-singular [illegible] by [illegible] respectively. Then by [illegible] and [illegible]

[illegible] (21)

[illegible] (22)

Introducing [illegible] and [illegible] where $j =$ [illegible] by the formula

[illegible] (23)

we have

[illegible]

i. e. in general

$$[a_{p,q}, b_{r,s}] = -(a+b)_{\rho[-(p+r)], \rho[-(q+s)]}. \tag{24}$$

From this follows

$$\begin{aligned}[a_{p,q}, b_{r,s}, c_{t,u}, d_{v,w}] &= [-(a+b)_{\rho[-(p+r)], \rho[-(q+s)]}, -(c+d)_{\rho[-(t+v)], \rho[-(u+w)]}] \\ &= (a+b+c+d)_{\rho(p+r+t+v), \rho(q+s+u+w)}; \end{aligned} \tag{25}$$

and hence, by a process similar to that employed in the previous cases,—

IV. *The residual of any* $3\mathrm{n}-1$ *indices on a non-singular cubic is their negative sum with first and second suffices equal to the minimum positive residues* (*mod.* 3) *of the negative sums of their first and second suffices, respectively; and the coresidual of any* $3\mathrm{n}+1$ *indices on such a cubic is their sum with first and second suffices equal to the minimum positive residues* (*mod.* 3) *of the sums of their first and second suffices, respectively.*

An extension to the sum of $3n-1$, the negative sum of $3n+1$, and the positive and negative sums of $3n$ indices, with the various combinations of suffices, may be made here, as in the previous cases.

Compound Derivation.

The problem of compound derivation is to determine the point which is derived from a given derivative just as a certain other derivative is obtained from the initial. For instance, to determine the a of b is to determine the index which is derived from b just as a is from 1, where a and b may have any suffices in accordance with the notation already employed. In forming the derivatives of b, evidently the same operations (additions and changes of sign) are performed as in forming the derivatives of 1; the only difference is, that in the former case they are applied to multiples of b, but in the latter case to the same multiples of 1. Hence the numerical value of a of b is a times the numerical value of b, i. e. is ab, to which, if necessary, proper suffices must be given; and the problem really reduces to that of finding the suffices of the compound derivative when those of the components are known.

In the case of a cuspidal cubic there is only one point of inflexion, no indices are necessary, and, for all values of a and b,

$$a \text{ of } b = ab. \tag{26}$$

In the case of an acnodal or crunodal cubic only one set of suffices is necessary, and evidently

$$a_0 \text{ of } b_\rho = (ab)_\rho, \tag{27}$$

if s is of the form [illegible] $n - 1$ [illegible] integer, and [illegible] either [illegible] If s is of the form [illegible] $n - 1 - t$ [illegible] the form [illegible] $n -$ [illegible] and hence

$$[illegible] = [illegible] = [illegible] = \varpi_{[illegible]}$$ [illegible]

If s [illegible] the form [illegible] $n -$ [illegible] $- 1$ [illegible] the form [illegible] $n - 1$ and $- 1$ of the form [illegible] $n -$ [illegible] and hence

$$[illegible] = [illegible] - 1 [illegible] - 1 [illegible]$$

$$= [illegible] - 1 [illegible] = \varpi_{[illegible]}$$ [illegible]

so that for all values of [illegible] and [illegible]

$$[illegible] = [illegible]$$ [illegible]

Hence for all values of [illegible] and [illegible]

$$[illegible] = [illegible] = [illegible] = \varpi_{[illegible]}$$ [illegible]

In the case of the [illegible] there are two sets of suffixes, and evidently

$$[illegible] = [illegible]$$ [illegible]

if s is of the form [illegible] $n - 1$, while if t is of the form [illegible] $n - 1$ and therefore $- t$ of the form [illegible] $n - 1$

$$[illegible] = [illegible] = [illegible] = \varpi_{[illegible]}$$ [illegible]

but if s is of the form [illegible] $n -$ [illegible] $- 1$ [illegible] the form [illegible] $n - 1$ and $- 1$ of the form [illegible] $n - 1$ and hence

$$[illegible] = [illegible] - 1 [illegible] - [illegible]$$

$$= [illegible] - 1 [illegible] = \varpi_{[illegible]}$$ [illegible]

so that for all values of [illegible] and [illegible]

$$[illegible] = [illegible]$$ [illegible]

From this follows for all values of [illegible] and [illegible]

$$[illegible] = [illegible]$$

$$= [illegible]$$ [illegible]

$$= [illegible]$$

The series of [illegible] with [illegible] of the form [illegible] $n - 1$ without suffix, with suffix [illegible] with suffixes [illegible] the series of [illegible] whose determinants [illegible] is called by Professor Sylvester the "natural scale" of

derivatives of the initial. Such a system is a closed system, i. e. the connective of any two natural derivatives of a point is also a natural derivative of that point. The natural scale taken with either series of indices of the form $3m-1$ and a certain series of indices of the form $3m$ constitutes a closed system. For example, the three series $(3m+1)_0$, $(3m-1)_1$, $(3m)_2$ constitute a closed system, as also $(3m+1)_{00}$, $(3m-1)_{12}$, $(3m)_{21}$. In fact, if p, q, r are the numbers 0, 1, 2 in any order, and s, t, u the same numbers in any order, then $(3m+1)_{p,s}$, $(3m-1)_{q,t}$, $(3m)_{r,u}$ constitute a closed system.

If real derivatives alone are to be considered, the system of indices without a suffix may be employed for crunodal as well as cuspidal cubics, and that with one suffix for non-singular as well as acnodal cubics. Indeed, if account is taken only of the closed system $(3m+1)_{p,s}$, $(3m-1)_{q,t}$, $(3m)_{r,u}$, there is no necessity for writing the indices, of which the form $3m+1$, $3m-1$, or $3m$ gives sufficient indication.

The system without suffices is to be regarded as included in that with one suffix, which is itself a special case of that with two suffices; and it will be convenient in the sequel to give every index two suffices, supplying 0 in place of each missing suffix; thus what I have heretofore denoted by a will now be denoted by a_{00}, and that heretofore denoted by a_p will be denoted by $a_{p,0}$ or $a_{0,p}$. Some agreement must be made as to the manner of supplying the missing suffix when only one is expressed, and this will affect in some degree the application of the theory of suffices to that of parameters, about which I shall presently say more.

The system above given without a suffix, as applied to non-singular cubics, is the only one explicitly treated by Professor Sylvester, who actually expresses the co-ordinates of such derivatives as rational algebraic functions, and from these expressions proves that the degree of the co-ordinates of the a^{th} derivative of any initial in the co-ordinates of the initial is the square of the numerical value of a,* and from this he infers that the number of a^{th} subderivatives of any point of the cubic (i. e. the number of points of which the given point may be considered as derivative with the index a) is a^2. This theorem is only true of non-singular cubics, and the proof seems to be wanting that every point obtained by equating to given values the co-ordinates of the derivative in terms of the initial is a point of the cubic. In so far as the theorem is true it holds also in the notation of this paper, inasmuch as the absolute numerical value of the index of any given derivative is the same in my notation as in Professor Sylvester's. This missing step is made good in the sequel.

* See pages 184–189.

Application of the Theory of Indices to that of Parameters.

The co-ordinates of any point of a cubic can be expressed as functions (rational or irrational) of a single parameter, which functions are either non-periodic, singly periodic, or doubly periodic; * so that we can classify cubics as *non-periodic* (cuspidal), *singly periodic* (acnodal or crunodal), and *doubly periodic* (non-singular). The non-periodic functions are algebraic; the singly periodic either trigonometric with a real period $\ominus$,† or exponential with an imaginary period $2i\ominus$; and the doubly periodic functions are elliptic with a real period $2K$ and an imaginary period $2iK'$.‡

Let ω, ω' denote the periods of the cubic, of which, say for convenience, ω is real and ω' is imaginary. Let also (μ) denote the point to which corresponds the value μ of the parameter, and let the sign $\equiv$, if the modulus is not expressed, denote equality, congruity (mod. ω), congruity (mod. ω'), or congruity (mod. ω, ω'), according as the curve is non-periodic, singly periodic with real period ω, singly periodic with imaginary period ω', or doubly periodic with real period ω and imaginary period ω'. Then the condition that three points (μ), (μ'), (μ'') shall be collinear is

$$\mu + \mu' + \mu'' \equiv 0, \tag{37}$$

i.e. the parameters of collinear points satisfy a congruence similar to the equations satisfied by the indices of three collinear derivatives of a common initial. Hence, if a is any integer of the form $3m + 1$,

$$a_{00} \text{ of } (\mu) = (a\mu). \tag{38}$$

---

* This representation of the singular cubics seems to be due to Salmon (Higher Plane Curves, 1st ed., 1852, Arts. 177, 178, and 183); and that of the non-singular cubics to Clebsch (Ueber einen Satz von Steiner und einige Punkte der Theorie der Curven dritter Ordnung, Crelle's Journal, Vol. LXIII., 1864). For a more thorough and systematic treatment, see Durège, Ueber fortgesetztes Tangentenziehen an Curven dritter Ordnung mit einem Doppel- oder Rückkehrpunkte, Math. Annalen, Vol. I. pp. 509 – 532; and Clebsch, Vorlesungen über Geometrie (herausgegeben von Lindemann), Vol. I. pp. 602 – 660. For further references, see Bibliography at the end of this article.

† I use the sign $\ominus$ to denote the ratio of the circumference of a circle to its diameter, usually represented by π, and the reversed sign $\mathfrak{S}$ to denote the base of the natural system of logarithms.

‡ With the usual notation K and K' are the complete integrals of the first kind corresponding to the modulus k and its complementary k'.

The simplest representation seems to be the following, in which x, y, z denote the homogeneous co-ordinates of any point of the cubic: —

For a cuspidal cubic, $x : y : z = \mu : 1 : \mu^3$,
For an acnodal cubic, $x : y : z = \sin\mu : \cos\mu : \sin^3\mu$,
For a crunodal cubic, $x : y : z = \mathfrak{S}^{\mu} : \mathfrak{S}^{2\mu} : 1 - \mathfrak{S}^{3\mu}$,
For a non-singular cubic, $x : y : z = \text{sn}\,\mu : \text{cn}\,\mu\,.\,\text{dn}\,\mu : \text{sn}^3\mu$.

From (37) follows that the parameters corresponding to the inflexions are the solution of the congruence

$$3\mu \equiv 0,$$

i. e. for a non-periodic cubic, $\mu = 0;$

for a singly periodic cubic, $\mu \equiv 0, \frac{1}{3}\omega, \frac{2}{3}\omega;$

for a doubly periodic cubic, $\mu \equiv 0 \quad , \frac{1}{3}\omega \qquad , \frac{2}{3}\omega \qquad ,$

$$\frac{1}{3}\omega', \frac{1}{3}\omega + \frac{1}{3}\omega', \frac{2}{3}\omega + \frac{1}{3}\omega',$$

$$\frac{2}{3}\omega', \frac{1}{3}\omega + \frac{2}{3}\omega', \frac{2}{3}\omega + \frac{2}{3}\omega'.$$

So that, in general, say,

$$0_{p,q} = p.\tfrac{1}{3}\omega + q.\tfrac{1}{3}\omega'. \tag{39}$$

From (24), (38), and (39) follows

$$\begin{aligned} a_{p,q} \text{ of } (\mu) &= [-a_{0,0} \text{ of } (\mu), 0_{\rho(-p),\rho(-q)}] = [(-a\mu), (-p.\tfrac{1}{3}\omega - q.\tfrac{1}{3}\omega')] \\ &= (a\mu + p.\tfrac{1}{3}\omega + q.\tfrac{1}{3}\omega'). \end{aligned} \tag{40}$$

From (39) and (40) follows that the first suffix of any index refers to the real and the second to the imaginary period; hence, in supplying a missing suffix, the new suffix 0 must be made the second or first according as the existing period is real or imaginary.

From the expressions for the co-ordinates given in the third footnote on page 368, it is evident that the necessary and sufficient condition for the reality of the point corresponding to a value μ of the parameter is

$$\mu \equiv \nu \,(\text{mod.}\ \tfrac{1}{2}\omega'), \tag{41}$$

where ν is real, which is equivalent to the condition that either μ is real or its imaginary part is half the imaginary period; if there is no imaginary period the parameter is real.

From (40) it is evident that any point whose parameter differs from an integral multiple of the parameter of a given point by integral multiples of the periods of the inflexions (which are $\frac{1}{3}\omega$ and $\frac{1}{3}\omega'$) is a rational derivative of the given point. In this sense *the theory of rational derivatives is the theory of commensurable parameters.*

The only real derivatives of a real point are evidently, from (40) and (41), those whose index has its second suffix 0, i. e. is of the form $a_{p,0}$, and every such derivative of a real point is real.

If (λ) is the $a_{p,q}$ of (μ),

$$\lambda \equiv a\mu + \tfrac{1}{3}(p\omega + q\omega'), \tag{42}$$

from which follows

$$\mu \equiv \frac{3\lambda + (3m - p)\,\omega + (3m' - q)\,\omega'}{3a}, \tag{43}$$

where each of the numbers m, m' is any integer less than $\pm a$* (including 0). The number of such sub-$a_{p,q}$'s of any point λ is thus evidently 1, $\pm a$, or a^2, according as the curve is non-periodic, singly periodic, or doubly periodic. For the doubly periodic or non-singular cubics this is Professor Sylvester's "law of squares." From the fact that the parameters given by (43) are all incongruent follows that the number of subderivatives of a given index is in *all* cases that just stated.

Moreover, it is evident from the formula that, *in general,* a point (μ) will not be at once a sub-$a_{p,q}$ and a sub-$b_{r,s}$ of the same point (λ), if b, r, s have other values than a, b, q respectively. There will, however, be points on the cubic, of which a subderivative with given index $b_{r,s}$ coincides with a subderivative with given index $a_{p,q}$, if a and b are different. To determine them, suppose (λ) to be a point of which a sub-$a_{p,q}$, say (μ), coincides with a sub-$b_{r,s}$, where for convenience we will suppose $a > b$; then (λ) is at once an $a_{p,q}$ of (μ) and a $b_{r,s}$ of (μ), hence

$$\lambda = a\mu + \tfrac{1}{3}(p\omega + q\omega') = b\mu + \tfrac{1}{3}(r\omega + s\omega'), \tag{44}$$

and

$$(a - b)\,\mu = \tfrac{1}{3}\left[(r - p)\,\omega + (s - q)\,\omega'\right], \tag{45}$$

$$\mu = \frac{(3m + r - p)\,\omega + (3m' + s - q)\,\omega'}{3(a - b)}, \tag{46}$$

$$\lambda = \frac{(3ma + ra - pb)\,\omega + (3m'a + sa - qb)\,\omega'}{3(a - b)}, \tag{47}$$

where each of the numbers m, m' has any value from 0 to $a - b - 1$ inclusive, from which it is evident that the number of such points (μ) is $1, (a - b)$, or $(a - b)^2$, according as the curve is non-periodic, singly periodic, or doubly periodic, and the number of such points (λ) is $1, \frac{a - b}{\delta}$, or $\frac{(a - b)^2}{\delta^2}$, where δ is the greatest common divisor of a and b.

Formula (44) shows that, if $b = a$, then $r = p$ and $s = q$, as is evident from the fact that any $a_{r,s}$ is obtained from the $a_{p,q}$ by collineation first with a certain inflexion $0_{\rho(r-p),\rho(s-q)}$ and then with the inflexion 0_{00}, and the two points cannot coincide unless $0_{\rho(r-p),\rho(s-q)}$ coincides with 0_{00}, i. e. $r = p$ and $s = q$.

Equation (45) shows that if the $a_{p,q}$ of (μ) coincides with the $b_{r,s}$ of (μ), then

* Here and in the following pages I employ $\pm a$ to denote the absolute (positive) value of a.

the $(a-b)$ (with any suffices) of (μ) is a certain inflexion, and especially the $(a-b)_{\rho(p-r),\rho(q-s)}$ of (μ) is the inflexion 0_{00}.

Equation (42) enables us to determine the condition that of two points given by their parameters, one (λ) is a derivative of the other (μ). The condition is evidently that of the possibility of the determination of three integers a, p, q to satisfy the congruence (42). It will be convenient to consider λ under the form $\lambda_1\omega+\lambda_2\omega'$, and μ under the form $\mu_1\omega+\mu_2\omega'$, — a representation which is unambiguous if λ_1, λ_2, μ_1, μ_2 are real, since ω and ω' are respectively real and purely imaginary. The condition is then involved in these two: —

$$\begin{aligned}\lambda_1 &\equiv a\mu_1+\tfrac{1}{3}p\,(\text{mod. }1),\\ \lambda_2 &\equiv a\mu_2+\tfrac{1}{3}q\,(\text{mod. }1),\end{aligned}\tag{48}$$

i. e. $\frac{1}{3}(3a\mu_1-3\lambda_1+p)$ and $\frac{1}{3}(3a\mu_2-3\lambda_2+q)$ are both integers, conditions which are certainly impossible unless λ_1 and μ_1, as also λ_2 and μ_2, are commensurable, in the sense that one can be expressed rationally in terms of the other and of absolute rationals; the most interesting case is that in which μ_1 and μ_2, and consequently λ_1 and λ_2, are rational quantities.

Periodic Points or Self-Derivatives.

A general problem which has many special forms of interest is this: to find a point (μ) whose $a_{p,q}$ coincides with it. The solution is given by

$$\begin{aligned}a\mu+\tfrac{1}{3}(p\omega+q\omega') &\equiv \mu,\\ (a-1)\mu &\equiv -\tfrac{1}{3}(p\omega+q\omega'),\\ \mu &\equiv \frac{(3m-p)\,\omega+(3m'-q)\,\omega'}{3(a-1)},\end{aligned}\tag{49}$$

in which, to obtain all the different points (μ) for a given index $a_{p,q}$, to each of the integers m, m' must be given any $\pm(a-1)$ successive values. The number of such different points (μ) for the index $a_{p,q}$ is therefore 1, $\pm(a-1)$, or $(a-1)^2$, according as the cubic is non-periodic, singly periodic, or doubly periodic. On the non-periodic curve the only self-derivative is the inflexion, which corresponds to every index, and I therefore omit the further consideration of this case. Equation (49) may also be written

$$\mu\equiv\frac{(-3m+p)\,\omega+(-3m'+q)\,\omega'}{3[-(a-2)-1]},\tag{50}$$

i.e. any self-$a_{p,q}$ is also a self-$[-(a-2)]_{\rho(-p),\rho(-q)}$, so that the indices go

together in pairs of *conjugates* (excepting the index $1_{0.0}$, which we need not consider, as every point is its own $1_{0.0}$) such that to each index of any pair of *complementary* indices, say, correspond the same self-derivatives. It is to be noticed that one of the indices of every pair is positive, so that the self-derivatives *may* be classed according to the positive index. However, in the formula (49) there occurs only $a - 1$, which has the same absolute value for any index a as for its conjugate. In what follows I therefore assume $a - 1$ to mean this absolute positive value, without writing the double sign, and take a either positive or negative.

If $3m - p$, $3m' - q$, and $a - 1$ in (49) have a common factor, the corresponding point (μ) may also be obtained for a less value of a (more exactly, for a less value of $a - 1$) with proper suffices. We may impose upon a, p, q any conditions we please (for instance, that a shall be of the form $3i + 1$, and $p = 0$, $q = 0$), which may be regarded as conditions imposed upon $3m - p$, $3m' - q$, and $a - 1$. Any point which is a self-derivative with the index $a_{p,q}$, but not a self-derivative with a less index, subject to the given conditions (say ψ) will be said to *belong to the index* $a_{p,q}$ (*conditions* ψ). The number of self-derivatives belonging to the index $a_{p,q}$ (conditions ψ) is evidently the number of pairs of numbers, the first of the form $3m - p$ and the second of the form $3m' - q$, neither greater than $3(a - 1)$, which do not *both* contain any divisor δ of $a - 1$ (other than 1) such that the quotients $\frac{3m - p}{\delta}$, $\frac{3m' - q}{\delta}$, and $\frac{a - 1}{\delta}$ also satisfy the given conditions. The pairs of values m, m' which are to be excluded are those which satisfy the congruences

$$3m \equiv p \,(\text{mod. } \delta) \qquad \text{and} \qquad 3m' \equiv q \,(\text{mod. } \delta),$$

where δ is some divisor of $a - 1$ satisfying a certain condition κ, which for convenience I assume to include the condition just stated that δ is a divisor of $a - 1$. With this proviso, κ is the condition that the quotients of $3m - p$, $3m' - q$, and $a - 1$, by δ, shall satisfy the conditions ψ.

If δ is not a multiple of 3, the number of numbers m not greater than $a - 1$ (subject to no condition) and satisfying the first of these congruences is $\frac{a - 1}{\delta}$, which is also the number of numbers not greater than $a - 1$ and divisible by δ; and the number of values of m' satisfying the second congruence is the same. Hence, if $a - 1$ is not a multiple of 3, so that it has no divisor δ which is such a multiple, the number of numbers of the form $3m - p$, not greater than $3(a - 1)$, which have no divisor satisfying the condition κ, is $\overset{a-1}{\underset{1}{T}}[\bar{\text{c}}\,(\widehat{a - 1}) \,.\, \kappa]$,* and the

* See note at the end of this article for definitions of these totients.

number of pairs of numbers, the first of the form $3m - p$ and the second of the form $3m' - q$, neither greater than $3(a-1)$, which have no common divisor satisfying the condition κ, is $\overset{a-1}{\underset{1}{T}}\,[\breve{\bar{\mathrm{O}}}\,(\widehat{a\dot{-}1})\,.\,\kappa]^2$.

If δ is a multiple of 3, and p is not 0, there is no value of m which satisfies the congruence $3m \equiv p \pmod{\delta}$. Hence, if $a-1$ has any divisors which are multiples of 3 satisfying the condition κ, and if p is 1 or 2, the number of numbers of the form $3m - p$, not greater than $3(a-1)$, which have no divisor satisfying the condition κ, is $\overset{a-1}{\underset{1}{T}}\,[\bar{\mathrm{O}}\,(\widehat{a-1})\,.\,\kappa\,.\,\bar{0}_3]$. Similarly, if $a-1$ has any divisors which are multiples of 3 satisfying the condition κ, and if either p or q (or both) is different from 0, the number of pairs of numbers, the first of form $3m - p$ and the second of the form $3m' - q$, neither greater than $3(a-1)$, which have no common divisor satisfying the condition κ, is $\overset{a-1}{\underset{1}{T}}\,[\breve{\bar{\mathrm{O}}}\,(\widehat{a\dot{-}1})\,.\,\kappa\,.\,\bar{0}_3]^2$.

If δ is a multiple of 3, and $p = 0$, the number of numbers of the form $3m$, not greater than $3(a-1)$, which have in common with $a-1$ the divisor δ is $\frac{3(a-1)}{\delta}$, which is the number of numbers not greater than $a-1$ having in common with $a-1$ the divisor $\frac{1}{3}\delta$; in fact, if $3m$ contains δ, a multiple of 3, then m will contain $\frac{1}{3}\delta$, and conversely. The number of numbers of the form $3m$, not greater than $3(a-1)$, which have no divisor satisfying the condition κ, is $\overset{a-1}{\underset{1}{T}}\,[\bar{\mathrm{O}}\,(\widehat{a-1})\,.\,\kappa\,.\,(\bar{0}_3, \frac{1}{3}0_3)]$;* and the number of pairs of numbers, each of the form $3m$, neither greater than $3(a-1)$, which have no common factor satisfying the condition κ, is $\overset{a-1}{\underset{1}{T}}\,[\breve{\bar{\mathrm{O}}}\,(\widehat{a\dot{-}1})\,.\,\kappa\,.\,(\bar{0}_3, \frac{1}{3}0_3)]^2$.

From what precedes it is evident that the number of self-derivatives belonging to the index $a_{p,q}$ (conditions ψ) is given by the following table:—

		Singly Periodic Cubic.	Doubly Periodic Cubic.
a not of the form $3i+1$		$\overset{a-1}{\underset{1}{T}}[\bar{\mathrm{O}}\,(\widehat{a-1})\,.\,\kappa]$	$\overset{a-1}{\underset{1}{T}}[\breve{\bar{\mathrm{O}}}\,(\widehat{a\dot{-}1})\,.\,\kappa]^2$
a of the form $3i+1$	p and q not both 0	$\overset{a-1}{\underset{1}{T}}[\bar{\mathrm{O}}\,(\widehat{a-1})\,.\,\kappa\,.\,\bar{0}_3]$	$\overset{a-1}{\underset{1}{T}}[\breve{\bar{\mathrm{O}}}\,(\widehat{a\dot{-}1})\,.\,\kappa\,.\,\bar{0}_3]^2$
	p and q both 0	$\overset{a-1}{\underset{1}{T}}[\bar{\mathrm{O}}(\widehat{a-1}).\kappa.(\bar{0}_3, \frac{1}{3}0_3)]$	$\overset{a-1}{\underset{1}{T}}[\breve{\bar{\mathrm{O}}}(\widehat{a\dot{-}1})\,.\,\kappa\,.(\bar{0}_3, \frac{1}{3}0_3)]^2$

(51)

The number of *real* self-derivatives belonging to the index $a_{p,q}$ (conditions ψ) is found by taking into account the condition (41), which being applied to (49) gives

$$2(3m' - q) \equiv 0\ [\text{mod. } 3(a-1)].$$

* For convenience, I have used here $\bar{\mathrm{O}}\,(\widehat{a-1})\,.\,\kappa\,.\,\frac{1}{3}0_3$ to denote "does not contain one third of any divisor of $a-1$ of the form $3i$ which satisfies the condition κ."

Now $3m' - q$ is never greater than $3(a-1)$; hence the only values of m' which will correspond to real points are those for which

$$3m' - q = 3(a-1) \text{ or the congruent value } 0, \text{ and } 3m' - q = \tfrac{3}{2}(a-1),$$

neither of which can be satisfied unless $q = 0$; and if $q = 0$, the solutions are, respectively, $m' = a - 1$ or 0, and $m' = \frac{1}{2}(a-1)$, which latter solution exists only when a is odd. Hence, whatever the given conditions, no *real* self-derivative belongs to any index of the type $a_{p,1}$ or $a_{p,2}$; and to the index $a_{p,0}$ belong both or only the first of the self-derivatives

$$\mu = \frac{3m-p}{3(a-1)}\,\omega \qquad \text{and} \qquad \mu = \frac{3m-p}{3(a-1)}\,\omega + \tfrac{1}{2}\omega', \tag{52}$$

according as a is odd or even, in each of which all values not greater than $a - 1$ (or any natural succession of $a - 1$ values) are to be assigned to m, rejecting only those for which $3m - p$ has in common with $a - 1$ a factor satisfying the conditions κ. The number of *real* self-derivatives on a doubly periodic cubic is then, by (51): —

$$\begin{array}{llll}
\overset{a-1}{\underset{1}{T}}[\bar{\text{o}}(\widehat{a-1}).\kappa] & \text{if } a \text{ is of the form} & & 6i \text{ or } 6i+2, \\
2\overset{a-1}{\underset{1}{T}}[\bar{\text{c}}(\widehat{a-1}).\kappa] & \text{“} & \text{“} & 6i+3 \text{ or } 6i+5, \\
2\overset{a-1}{\underset{1}{T}}[\bar{\text{o}}(\widehat{a-1}).\kappa.\bar{0}_3] & \text{“} & \text{“} & 6i+1 \text{ and } p \text{ is } 1 \text{ or } 2, \\
\overset{a-1}{\underset{1}{T}}[\bar{\text{o}}(\widehat{a-1}).\kappa.\bar{0}_3] & \text{“} & \text{“} & 6i+4 \text{ and } p \text{ is } 1 \text{ or } 2, \\
2\overset{a-1}{\underset{1}{T}}[\bar{\text{o}}(\widehat{a-1}).\kappa.(\bar{0}_3, \tfrac{1}{3}0_3)] & \text{“} & \text{“} & 6i+1 \text{ and } p \text{ is } 0, \\
\overset{a-1}{\underset{1}{T}}[\bar{\text{o}}(\widehat{a-1}).\kappa.(\bar{0}_3, \tfrac{1}{3}0_3)] & \text{“} & \text{“} & 6i+4 \text{ and } p \text{ is } 0;
\end{array} \tag{53}$$

i. e. the number of *real* self-derivatives (suffices p, q) on a doubly periodic cubic is the same as or double the number of self-derivatives (suffices p, q) on a singly periodic cubic, according as a is even or odd.

On a singly periodic cubic with real period only the first of equations (52) is to be employed, and all the self-derivatives are real. The only real self-derivatives on a singly periodic cubic with imaginary period are those points for which

$$\mu = 0 \qquad \text{and} \qquad \mu = \tfrac{1}{2}\omega',$$

the former of which corresponds (in an improper sense) to every index, and the latter (in the same sense) to every odd index; the latter is the point of contact

of the tangent which can be drawn from the real inflexion to touch the curve elsewhere.

If no conditions ψ are given, I say simply that the self-derivative (μ) *belongs to* the index $a_{p,q}$. If to m be assigned successively all values not exceeding $a-1$, and to p each of the values 2, 1, 0, the number $3m-p$ assumes successively all values not exceeding $3(a-1)$, and each value but once. Hence the number of self-derivatives belonging to the index $a_{p,q}$ is the number of pairs of numbers, the first of the form $3m-p$ and the second of the form $3m'-q$, neither greater than $3(a-1)$, which have no common divisor which divides also $a-1$; and this number is found from (51) by making the condition κ mean only "divisor of $a-1$," i. e. in the notation of the appended note

$$\kappa = (\widehat{a-1}). \tag{54}$$

In this classification, if a is of the form $3i+1$ and p and q are both 0, since there is no condition imposed upon the least divisors of $a-1$, the only least divisor which is a 0_3 is 3 itself; hence

$$\overset{a-1}{\underset{1}{T}}[\bar{0}\,\kappa\,.\,(\bar{0}_3\,,\tfrac{1}{3}\,0_3)] = \overset{a-1}{\underset{1}{T}}[\bar{0}\,\kappa\,.\,(\bar{0}_3\,,1)] = 0,$$

because it contains the vanishing factor $\left(1-\frac{1}{1}\right)$. Hence no self-derivative belongs to the index $a_{0,0}$, if a is of the form $3i+1$, and no condition is imposed. These results apply to the doubly periodic cubic; for the singly periodic cubic, the number of self-derivatives belonging to the index $a_{p,0}$ or $a_{0,q}$, according as the period is real or imaginary, is the number of numbers of the form $3m-p$ or $3m'-q$, not greater than $a-1$, which have no common divisor which divides $a-1$; and this number is also found from (51) by determining κ to satisfy (54) alone.

The first classification of self-derivatives under an imposed condition which I shall consider is that in which the suffices are given; thus, in accordance with the above convention, I shall speak of the self-derivatives belonging to the index a (suffices p, q); it is needless in this case to repeat the suffices. As an example of this kind of classification consider the point corresponding to $\mu = \frac{5}{6}\,\omega + \frac{2}{3}\,\omega'$, which may be written

$$\mu = \frac{(3\,.\,2-1)\,\omega + (3\,.\,2-2)\,\omega'}{3\,(3-1)} = \frac{(3\,.\,4-2)\,\omega + (3\,.\,3-1)\,\omega'}{3\,(5-1)} = \frac{3\,.\,5\,\omega + 3\,.\,4\,\omega'}{3\,(7-1)},$$

which belongs to the indices $3_{1,2}$ and $-1_{2,1}$ (no conditions); but in the present classification it belongs to the indices 3 and -3 (suffices 1, 2), to 5 and -1

(suffices 2, 1), and to 7 and -5 (suffices 00). The condition ψ that a self-derivative given by (49) shall belong to the index a (suffices p, q) is evidently that $3m-p$, $3m'-q$, and $a-1$ shall contain no common factor δ such that $3m-p$ and $3m'-q$ shall have, respectively, the same residues (mod. 3) as their quotients by δ. This constancy of residue is the condition κ. If p and q are not both 0, this condition is satisfied when δ is of the form $3i+1$, and only then, so that $\kappa = (\widehat{a-1}) . 1_3$; but if p and q are both 0, $\frac{3m}{\delta}$ will contain 3 when m contains δ, i.e. the condition ψ will be satisfied if m is a divisor of $a-1$; so that $\kappa = (\widehat{a-1})$.

In this classification, therefore, the number of self-derivatives belonging to the index a (suffices p, q) is as follows:—

p and q.	Singly Periodic with Real Period.	Doubly Periodic.
Not both 0	$\overset{a-1}{\underset{1}{T}}[\bar{\text{O}}(\widehat{a-1}) . 1_3]$	$\overset{a-1}{\underset{1}{T}}[\breve{\bar{\text{O}}}(\widehat{a \dot{-} 1}) . 1_3]^2$
Both 0	$\tau(a-1)$	$\tau^2(a-1)$

(55)

A very important classification of self-derivatives is that according to the least index having a given residue (mod. 3) with given suffices, say ($\equiv a$, mod. 3; suffices p, q). It is evident that, unless a is of the form $3i+1$, and p and q are both 0, the number of self-derivatives belonging to the index a ($\equiv a$, mod. 3; suffices p, q) is $\overset{a-1}{\underset{1}{T}}[\bar{\text{o}}(\widehat{a-1}) . 1_3]$ or $\overset{a-1}{\underset{1}{T}}[\breve{\bar{\text{o}}}(\widehat{a \dot{-} 1}) . 1_3]^2$, according as the cubic is singly or doubly periodic. If a is of the form $3i+1$, and p and q are both 0, the condition κ is that the divisor δ shall be such that the quotients $\frac{3m}{\delta}$, $\frac{3m'}{\delta}$, $\frac{a-1}{\delta}$ are all multiples of 3, as the first two are if δ is a divisor of m and m', and $\frac{a-1}{\delta}$ will be if 3 occurs as a factor to a higher power in $a-1$ than in δ; if $a-1$ contains 3 to a higher power than the first, every factor of $a-1$ will have a divisor δ containing 3 to a less power than occurs in $a-1$; but if $a-1$ contains only the first power of 3, then every factor of $a-1$, excepting only 3 itself, will have a divisor δ which contains 3 to a less power than occurs in $a-1$, but 3 will not have such a divisor. Hence, if a is of the form $9i+1$, and p and q are both 0, the number of self-derivatives belonging to a ($\equiv 1$, mod. 3; suffices 00) is $\tau(a-1)$ for a singly periodic, and $\tau^2(a-1)$ for a doubly periodic cubic; and if a is of the form $3i+1$ but not of the form $9i+1$, and p and q are both 0, the number of self-derivatives belonging to

a ($\equiv 1$, mod. 3; suffices 00) is $\overset{a-1}{\underset{1}{T}}[\bar{\text{o}}\,(\widehat{a-1})\,.\,\bar{0}_3]$ for a singly periodic, and $\overset{a-1}{\underset{1}{T}}[\breve{\bar{\text{o}}}\,(\widehat{a \dot{-} 1})\,.\,\bar{0}_3]^2$ for a doubly periodic cubic.

If p and q are both 0, and a is of the form $3\,i+1$ or $-3\,i+1$,* where i is a positive integer, the self-derivatives are *pertactile* points of the grade i (Sylvester, page 74). The condition that a point (μ) should be a pertactile point belonging to the grade i is

$$\mu = \frac{m\omega + m'\omega'}{3\,i}, \tag{56}$$

where neither m nor m' exceeds $3\,i$, and m, m', and i have no common divisor. If i is a multiple of 3, $3\,i$ contains no prime factors which i does not, and the number of such points is $\tau(3\,i)$ or $\tau^2(3\,i)$, according as the cubic is singly or doubly periodic. If i is not a multiple of 3, $3\,i$ contains one prime factor, viz. 3, which i does not, and hence those numbers m and m', which contain only the factor 3 in common with $3\,i$, do not correspond to pertactile points of lower grade, and the number of pertactile points belonging to the grade i is $\overset{3i}{\underset{1}{T}}[\bar{\text{o}}\,(\widehat{3\,i})\,.\,\bar{0}_3]$ or $\overset{3i}{\underset{1}{T}}[\breve{\bar{\text{o}}}\,(\widehat{3\,i})\,.\,\bar{0}_3]^2$, according as the cubic is singly or doubly periodic. From the values of these totients given in the appended note, it is easy to see that, whatever the value of i, the number of pertactile points belonging to the grade i is $3\,\tau(i)$ or $9\,\tau^2(i)$, according as the cubic is singly or doubly periodic, which last result agrees with that given by Professor Sylvester (page 76).

In general, if (μ) is a self-derivative belonging (conditions given) to the index $a_{p,q}$, it follows from (49) that

$$\begin{aligned}
(a-1)\,\mu &\equiv -\tfrac{1}{3}\,(p\omega + q\omega'), & -(a-1)\,\mu &\equiv \tfrac{1}{3}\,(p\omega + q\omega'), \\
a\mu &\equiv \mu - \tfrac{1}{3}\,(p\omega + q\omega'), & -a\mu &\equiv -\mu + \tfrac{1}{3}\,(p\omega + q\omega'), \\
(a-2)\,\mu &\equiv -\mu - \tfrac{1}{3}\,(p\omega + q\omega'), & -(a-2)\,\mu &\equiv \mu + \tfrac{1}{3}\,(p\omega + q\omega').
\end{aligned} \tag{57}$$

Now, according as a is of the form $3\,i+1$, $-3\,i+1$, $3\,i-1$, $-3\,i$, $3\,i$, or $-3\,i-1$, where i is a positive integer, the number $-(a-2)$, a, $-a$, $a-2$, $-(a-1)$, or $a-1$ is of the form $-3\,n+1$, where n is a positive integer, viz. $n=i$ unless a is of the form $-3\,i$ or $-3\,i-1$, and then $n=i+1$. That is, in the six cases, respectively,

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* As has been remarked in connection with (50), the self-derivatives belonging to the index $a_{p,q}$ belong also to the index $-(a-2)_{\rho(p),\rho(q)}$, so that the indices go together in pairs, thus: $3i+1$, $-3i+1$; $3i$, $-3i+2$; $3i+2$, $-3i$ are the forms of pairs of indices to which belong the same self-derivatives, and in the following I shall assume such a grouping without further mention. Especially is this grouping useful when p and q are both 0, for $\rho(0)=0$, so that the suffices of any index (suffices 00) are the same as of the complementary index.

$$(-3n+1)\,\mu \equiv \mu + \tfrac{1}{3}(p\omega + q\omega'),\ \mu - \tfrac{1}{3}(p\omega + q\omega'),\ -\mu + \tfrac{1}{3}(p\omega + q\omega'),$$
$$-\mu - \tfrac{1}{3}(p\omega + q\omega'),\ \tfrac{1}{3}(p\omega + q\omega'),\ -\tfrac{1}{3}(p\omega + q\omega'),$$

so that, if a curve of the order n be passed through $3n-1$ consecutive points on the cubic at (μ), i. e. have $(3n-1)$-point* contact with the cubic at (μ), the $(3n)^{\text{th}}$ intersection of such a curve with the cubic will be, in the six cases respectively, the $1_{p,q}$, $1_{\rho(p),\rho(q)}$, $-1_{p,q}$, $-1_{\rho(p),\rho(q)}$, $0_{p,q}$, $0_{\rho(p),\rho(q)}$ of (μ); in the first two cases, if p and q are both 0, the $(3n)^{\text{th}}$ intersection will be the point (μ) itself, and in the last two cases it is a constant point, viz. an inflexion. These latter cases appear as special forms of the problem: to determine the points (μ) at which a curve of the order n passing through a given point (λ) of the cubic may have $(3n-1)$-point contact, the solution of which is given by $-(3n-1)\,\mu = \lambda$, so that (μ) is a sub-$(-3n+1)_{00}$ of (λ).

To the subject of self-derivatives belongs the problem of the in- and exscribed k-laterals, which may be stated thus: to construct a polygon of k sides, each tangent to the cubic at its intersection with the preceding side; or, in other words, to determine a point on the cubic, whose k^{th} tangential coincides with it. It is evident that the index of the k^{th} tangential of 1_{00}, which is found by a repetition of formula (2), is $(-2)^k_{00}$; hence the condition that (μ) shall be a vertex of an in- and exscribed k-lateral is $(-2)^k\mu \equiv \mu$, i. e. $2^k\mu \equiv (-1)^k\mu$, or

$$\mu \equiv \frac{m\omega + m'\omega'}{2^k - (-1)^k}, \tag{58}$$

where m and m' have any values not exceeding $2^k - (-1)^k$; the number $\psi(k)$ of vertices of *proper* k-*laterals*, i. e. corresponding to k but to no divisor of k, is

$$\overset{2^k-(-1)^k}{\underset{1}{T}}\left[\bar{\circ}\left(2^k-(-1)^k\right).\kappa\right] \quad \text{or} \quad \overset{2^k-(-1)^k}{\underset{1}{T}}\left[\breve{\bar{\circ}}\left(2^k-(-1)^k\right).\kappa\right]^2,$$

for a singly or doubly periodic cubic respectively, where the condition κ is to be taken to mean that the quotient of $2^k - (-1)^k$ by the divisor, is still of the form $2^i - (-1)^i$. Now $2^i - (-1)^i$ is a divisor of $2^k - (-1)^k$ if i is a divisor of k, and only then. For a singly periodic cubic, then, if $1, d, d', \ldots, k$ are the different divisors of k,

$$\psi(1) + \psi(d) + \psi(d') + \ldots + \psi(k) = 2^k - (-1)^k; \tag{59}$$

therefore, if $\alpha, \beta, \gamma, \ldots$ are the different prime factors of k, and if for convenience $2^k - (-1)^k$ is represented by $f(k)$,

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* I make a distinction between k-point contact and k-tuple contact of two curves, the former implying k, and the latter $k+1$ consecutive common points, although I believe such a distinction is not usual.

$$\psi(k) = f(k) - \Sigma' f\left(\frac{k}{a}\right) + \Sigma'' f\left(\frac{k}{a\beta}\right) - \Sigma''' f\left(\frac{k}{a\beta\gamma}\right) + \ldots, \qquad (60)$$

where $\Sigma', \Sigma'', \Sigma''', \ldots$ denote summation with respect to all the factors $a, \beta, \gamma, \ldots$, their binary, ternary, etc., products (for a proof of the theorem that (59) implies (60), see Dirichlet's *Zahlentheorie*, edited by Dedekind, § 138, or Bachmann's *Kreistheilung*, pp. 8–11); in other words,*

$$\psi(k) = [2^{(\,)} - (-1)^{(\,)}]\,\tau(k) = [2^{(\,)}]\,\tau(k). \qquad (61)$$

The equality of the second and third members of this equation follows from the fact that, in the prime totient of k, according to which the sum is taken in (61), the number of positive even terms is the same as that of negative even terms, and the number of positive odd terms is the same as that of negative odd terms,† so that

$$[(-1)^{(\,)}]\,\tau(k) = 0.$$

If the cubic is doubly periodic, (59) is to be replaced by

$$\psi(1) + \psi(d) + \psi(d') + \ldots + \psi(k) = \left(2^k - (-1)^k\right)^2, \qquad (62)$$

and hence follows, by virtue of the theorem above cited,

$$\psi(k) = f^2(k) - \Sigma' f^2\left(\frac{k}{a}\right) + \Sigma'' f^2\left(\frac{k}{a\beta}\right) - \Sigma''' f^2\left(\frac{k}{a\beta\gamma}\right) + \ldots, \qquad (63)$$

where $f^2(k), \ldots$, are the squares of $f(k), \ldots$, i. e.

$$\psi(k) = [2^{(\,)} - (-1)^{(\,)}]^2\,\tau(k) = [(-2)^{2(\,)} - 2(-2)^{(\,)}]\,\tau(k). \qquad (64)$$

From (58) it is evident that $k = 1$ gives all the inflexions, which are also *improper* solutions for every value of k, since $2^k - (-1)^k$ is divisible by 3, whatever value k may have.

Since $2^k - (-1)^k$ is necessarily odd, and hence $\frac{1}{2}[2^k - (-1)^k]$ cannot be an integer, it follows from a comparison of (58) and (41) that the number of *real*

* See "Note on Totients" at the end of this paper.

† Namely, let n be the number of different prime factors of k. If k is odd, all its prime factors are odd, and all the terms of its prime totient are odd, 2^{n-1} of them being positive, and 2^{n-1} negative. If k is oddly even, i. e. contains 2 but not 4, one of the prime factors of k is 2, and its prime totient contains the factor $(2-1)$, and otherwise only odd factors (i. e. odd monomial factors and binomial factors of the form $a-1$, where a is odd), so that 2^{n-2} terms are positive and even, 2^{n-2} negative and even, 2^{n-2} positive and odd, 2^{n-2} negative and odd. If k is evenly even, i. e. contains 4, its prime totient contains the factor 2, and every term is even, 2^{n-1} being positive, and 2^{n-1} negative. If $k = 1$, $\tau(k) = 1$, and $[2^{(\,)} - (-1)^{(\,)}]\,\tau(1) = 3$; and if $k = 2$, $\tau(k) = 2 - 1$, $[2^{(\,)} - (-1)^{(\,)}]\,\tau(2) = 0$; which are the only exceptions.

vertices of proper k-laterals in- and exscribable to a singly periodic cubic with real period or a doubly periodic cubic is $[2^{(\,)}]\,\tau(k)$, but that there are no such real vertices on a non-periodic cubic or a singly periodic cubic with imaginary period, excepting the one real inflexion on each of the latter cubics. The number of proper k-laterals in- and exscribable to a cubic is then given by the following table (excluding inflexions).

Species of Cubic.	Real and Imaginary.	Real.
CUSPIDAL (non-periodic)	0	0
CRUNODAL (singly periodic with imaginary period)	$\frac{1}{k}[2^{(\,)}]\,\tau(k)$*	0
ACNODAL (singly periodic with real period)	$\frac{1}{k}[2^{(\,)}]\,\tau(k)$	$\frac{1}{k}[2^{(\,)}]\,\tau(k)$
NON-SINGULAR (doubly periodic)	$\frac{1}{k}[(-2)^{2(\,)}-2(-2)^{(\,)}]\,\tau(k)$	$\frac{1}{k}[2^{(\,)}]\,\tau(k)$†

(65)

As Professor Sylvester has remarked,‡ the number of proper k-laterals in- and exscribable to a non-singular cubic shows that $[2^{(\,)}-(-1)^{(\,)}]^2\,\tau(k)$ is always divisible by k; but the method here employed shows that, if a is any integer whatever, positive or negative, every point (μ) for which a^k of (μ) coincides with (μ) is given by

$$\mu=\frac{m\omega+m'\omega'}{a^k-1}; \tag{66}$$

hence, by the same reasoning as above,

$$[a^{(\,)}-1]\,\tau(k)=[a^{(\,)}]\,\tau(k)$$

and

$$[a^{(\,)}-1]^2\,\tau(k)=[a^{2(\,)}-2a^{(\,)}]\,\tau(k)$$

are divisible by k; in general, $[a^{(\,)}-1]^i\,\tau(k)$, where i is any positive integer, is

* Compare Rosenow, *Die Curven dritter Ordnung mit einem Doppelpunkte*, p. 41, and Durège, *Ueber fortgesetztes Tangentenziehen an Curven dritter Ordnung mit einem Doppel- oder Rückkehrpunkte*, Math. Annalen, Vol. I.

† Compare Harnack, *Ueber die Verwerthung der elliptischen Functionen für die Geometrie der Curven dritten Grades*, Math. Annalen, Vol. IX. p. 12, footnote.

‡ This Journal, Vol. II. p. 386.

also divisible by k, for it may be developed as a sum of terms of the form $C\,.\,[a^{\prime\prime}]^r\,\tau(k)$ (where C and r are integers, of which the latter is positive), each of which terms contains k. Thus, not only the last member of (64), but each of its terms, is divisible by k.

On page 75 Professor Sylvester has obtained a result which may be more explicitly stated thus: If any $(3a)_{p,q}$ of (λ) coincides with the inflexion $0_{p,q}$, then the $a_{r,s}$ of (λ) will coincide with some inflexion; namely, the $a_{r,s}$ of a^2 different sub-$(3a)_{p,q}$'s of (λ) will coincide with any given inflexion, r and s being given, and the $a_{r,s}$ of any given sub-$(3a)_{p,q}$ of (λ) will coincide with any given inflexion for some *one* set of values of r and s. We may solve, by the preceding methods, this more general problem:—

If the $(ab)_{p,q}$ of (λ) is a given inflexion, under what conditions will the $a_{r,s}$ of (λ) also be an inflexion, and what inflexion will it be?

Let the given inflexion be $\frac{1}{3}(\mu\omega+\mu'\omega')$, then

$$\lambda=\frac{(\mu-p+3m)\,\omega+(\mu'-q+3m')\,\omega'}{3ab}, \tag{67}$$

where each of the numbers m and m' has any ab successive values, and the

$$a_{r,s}\text{ of }(\lambda)=\frac{(\mu-p+3m+br)\,\omega+(\mu'-q+3m'+bs)\,\omega'}{3b}, \tag{68}$$

which is an inflexion if

$$\mu-p+3m\equiv0\quad\text{and}\quad\mu'-q+3m'\equiv0\ (\text{mod. } b),$$

and only then. If then b is not a multiple of 3, the values of m and m' satisfying these conditions will be found from

$$3m\equiv p-\mu\quad\text{and}\quad3m'\equiv q-\mu'\ (\text{mod. } b), \tag{69}$$

and the number of points (λ) found by substituting these values in the above expression for λ is a^2; but if b is a multiple of 3, the congruences for m and m' have solutions only when $\mu-p$ and $\mu'-q$ are also divisible by 3, i. e. (since each of the numbers μ, μ', p, q has the value 0, 1, or 2) only when $\mu=p$ and $\mu'=q$; and if these conditions are satisfied, then

$$m\equiv0\quad\text{and}\quad m'\equiv0\ (\text{mod. } \tfrac{1}{3}b), \tag{70}$$

which give $9a^2$ points (λ) satisfying the given conditions. In either case say $3m=p-\mu+tb$, $3m'=q-\mu'+ub$, then

$$a_{r,s}\text{ of }(\lambda)=\frac{(r+t)\,\omega+(s+u)\,\omega'}{3}, \tag{71}$$

which may be *any* inflexion depending on the values of t and u. If $b = 3$, $\mu = p$, $\mu' = q$, equations (70) show that m and m' may be any integers; this is the special case to which reference has been made above.

Note on Totients.

In the foregoing paper reference has been made to certain numbers called "totients." A *totient* is the number of things which satisfy certain conditions. These conditions may be of any nature, affirmative or negative. The things satisfying the prescribed conditions are called *totitive* to that condition, or simply *totitives*. Every science has its totitives, whose nature depends upon the subject matter of the science, and the determination of the totient or number of things satisfying any possible conditions constitutes a distinct branch of the science, which may itself be designated as the *totics* of the subject. For instance, we have *geometrical totics* (German, "Abzählende Geometrie"), the province of which is to determine the number of geometrical figures or curves satisfying certain conditions. I propose here to give an outline of a notation for *arithmetical* totients, and some formulæ belonging to *arithmetical totics*.

I use the following notation:—

$a, b, c, d, n, p, q, \alpha, \beta, \gamma$, denote integers.

κ, χ, are logical symbols denoting conditions satisfied.

r_d (read "an r to d") is a logical symbol denoting that division by d leaves the remainder r.

$\hat{a}$ denotes some (or any) divisor of a (other than 1).

ɔ denotes "contains as divisor," and is followed by the number contained.

Besides the usual arithmetical multiplication and addition, it is necessary in combining *conditions* to employ logical multiplication and addition, which I denote by (.) and (,) respectively, with these definitions:—

$\kappa \,.\, \chi$ denotes that κ and χ are both (separately) satisfied.

$\kappa \,,\, \chi$ denotes that either κ or else χ is satisfied.

The logical product of two or more *numbers* denotes their least common multiple.

Whenever it becomes necessary to indicate how far the force of a logical sign or symbol extends, I use parentheses in conformity with the usual convention in the case of the arithmetical signs. With respect to the symbol ɔ or ɔ̄, it is assumed that the force of each extends over the signs (.) and (,). A dash over any logical symbol indicates that the condition denoted by that symbol is not satisfied; thus, $\bar{r}_d$ denotes that division by d leaves a remainder different from r.

$\overline{\kappa\,,\,\chi}$ indicates that neither κ nor χ is satisfied.

$\overline{\kappa\,.\,\chi}$, that κ and χ are not both satisfied.

$\mathrm{c}\hat{n}$ denotes "contains some factor of n," and $\bar{\mathrm{c}}\hat{n}$ "contains no factor of n."

A condition, or logical product or sum of conditions standing by itself denotes any number satisfying the simple or compound condition.

$\overset{q}{\underset{p}{T}}[\kappa]$ denotes the totient to the condition κ within the limits p to q, i. e. the number of numbers from p to q, inclusive, which satisfy the condition κ.

$T[\overset{q}{\underset{p}{\kappa}}]\,[\overset{s}{\underset{r}{\chi}}]$ denotes the number of pairs of numbers, the first between p and q, inclusive, satisfying the condition κ, and the second between r and s, inclusive, satisfying the condition χ.

$\overset{q}{\underset{p}{T}}[\kappa]^k$ denotes the number of sets of k numbers, each satisfying the condition κ, and between p and q, inclusive, counting repetitions due to different permutations of the numbers of any set. This notation may be extended indefinitely.

It is evident that

$$T[\overset{q}{\underset{p}{\kappa}}]\,[\overset{s}{\underset{r}{\chi}}] = \overset{q}{\underset{p}{T}}[\kappa]\;\overset{s}{\underset{r}{T}}[\chi],$$

and

$$\overset{q}{\underset{p}{T}}[\kappa]^k = (\overset{q}{\underset{p}{T}}[\kappa])^k,$$

whenever κ is a self-existent condition, i.e. one which may be predicated or denied of each number in the set of k, without reference to any other. For example, the condition $\bar{\mathrm{c}}\hat{n}$ is self-existent, but $\bar{\mathrm{c}}\hat{n}$ is not.

Similarly $\overset{q}{\underset{p}{T}}[\bar{\kappa}]$ is the number of numbers within the limits p to q, which do not satisfy the condition κ.

In the case of totients corresponding to *sets of numbers* it may be necessary to express conditions which are imposed upon each number of the set, but not independently of the other numbers. For this purpose I use the following notation:—

$[\kappa]^k$ denotes a set of k numbers *each* satisfying the condition κ.

$[\bar{\kappa}]^k$ denotes a set of k numbers *neither* satisfying the condition κ.

$[\breve{\kappa}]^k$ denotes a set of k numbers *some* (at least one) satisfying the condition κ.

$[\breve{\bar{\kappa}}]^k$ denotes a set of k numbers *not all* (some not) satisfying the condition κ.*

* The notations $\breve{\kappa}$ and $\breve{\bar{\kappa}}$ are borrowed from Mr. Peirce's *Algebra of Logic*, this volume, page 22.

$\hat{a}$ denotes some (any) factor of a, the same for each number of the set in question (usually to be read "any common factor of a"). For $k = 1$, $[\kappa]$ and $[\breve{\kappa}]$ are identical, as are also $[\bar{\kappa}]$ and $[\breve{\bar{\kappa}}]$.

In general,

$$\overset{n}{\underset{1}{T}}[\kappa] + \overset{n}{\underset{1}{T}}[\bar{\kappa}] = n, \qquad \overset{n}{\underset{1}{T}}[\kappa]^k + \overset{n}{\underset{1}{T}}[\breve{\bar{\kappa}}]^k = n^k, \qquad \overset{n}{\underset{1}{T}}[\bar{\kappa}]^k + \overset{n}{\underset{1}{T}}[\breve{\kappa}]^k = n^k. \quad (72)$$

According to the notation just explained, $\overset{n}{\underset{1}{T}}[\bar{\circ}\hat{n}]$ denotes the number of numbers not greater than n and relatively prime to it; it may be called the *prime totient* of n, and represented, for the sake of brevity, by $\tau(n)$, a notation introduced by Professor Sylvester to replace the customary $\phi(n)$. Similarly $\overset{n}{\underset{1}{T}}[\breve{\bar{\circ}}\hat{n}]^k$, the number of sets of k numbers, neither greater than n, which do not *all* contain any one factor of n, may be called the k^{th} *prime* totient of n, and represented by $\tau^k(n)$; Professor Sylvester has called $\tau^2(n)$ the "quadritotient" of n.

The totients which I have had occasion to use in the foregoing paper are those expressing the number of numbers within certain limits which have no divisors in common with a given number n, which satisfy certain conditions, i. e. totients of the type

$$\overset{n}{\underset{1}{T}}[\bar{\circ}\hat{n} \,.\, \kappa], \qquad \overset{n}{\underset{1}{T}}[\breve{\bar{\circ}}\hat{n} \,.\, \kappa]^k.$$

In what follows I assume:—

$a, b, c, \ldots.,$ to be all the different *prime* factors of n, so that $n = a^\alpha b^\beta c^\gamma, \ldots.;$

$1, d, d', \ldots., n$ to be all the different divisors of n; and

$\delta, \delta', \delta'', \ldots.,$ to be a complete system of *least* divisors of n satisfying the condition κ. By a system of *least divisors* I mean a system of divisors of which no one is a multiple of any other. Then, evidently,

$$\overset{n}{\underset{1}{T}}[\circ d] = n\,\frac{1}{d}, \qquad \overset{n}{\underset{1}{T}}[\circ d]^k = n^k\,\frac{1}{d_k}, \quad (73)$$

$$\overset{n}{\underset{1}{T}}[\circ d \,.\, d'] = n\,\frac{1}{d \,.\, d'}, \qquad \overset{n}{\underset{1}{T}}[\circ d \,.\, d']^k = n^k\,\frac{1}{d^k\ d'^k}, \quad (74)$$

and hence follows, by (72),

$$\overset{n}{\underset{1}{T}}[\bar{\circ} d] = n\left(1 - \frac{1}{d}\right), \qquad \overset{n}{\underset{1}{T}}[\breve{\bar{\circ}} d]^k = n^k\left(1 - \frac{1}{d^k}\right). \quad (75)$$

The repetition of these formulæ gives

$$\left.\begin{aligned} \tau(n) &= \overset{n}{\underset{1}{T}}[\circ\hat{n}] = n\left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{b}\right)\left(1 - \frac{1}{c}\right)\ldots., \\ \tau^k(n) &= \overset{n}{\underset{1}{T}}[\breve{\circ}\hat{n}] = n^k\left(1 - \frac{1}{a^k}\right)\left(1 - \frac{1}{b^k}\right)\left(1 - \frac{1}{c^k}\right)\ldots., \end{aligned}\right\} \quad (76)$$

of which the first is given in all the text-books, and the second, for the particular case $k = 2$, is given by Professor Sylvester (page 76).

The same method gives also

$$\begin{aligned} \overset{n}{\underset{1}{T}}[\bar{\mathsf{O}}\hat{n}\,.\,\kappa] &= n\left(1-\frac{1}{\delta.}\right)\left(1-\frac{1}{\delta'.}\right)\left(1-\frac{1}{\delta''.}\right)\ldots., \\ \overset{n}{\underset{1}{T}}[\breve{\bar{\mathsf{O}}}\hat{n}\,.\,\kappa] &= n^k\left(1-\frac{1}{\delta^k.}\right)\left(1-\frac{1}{\delta'^k.}\right)\left(1-\frac{1}{\delta''^k.}\right)\ldots., \end{aligned} \tag{77}$$

in which, as is indicated by the dots, the products of factors δ, δ', δ'', ..., in the denominators of the expanded expressions are *logical* products, i. e the *least common multiples* of their factors. More generally, if δ, δ', δ'', ... is the complete system of least divisors of n for the condition κ, and δ_1, δ_1', δ_1'', ..., the complete system for the condition χ,

$$\begin{aligned} \overset{n}{\underset{1}{T}}[\bar{\mathsf{O}}(\hat{n}\,.\,\kappa)\,.\,\mathsf{O}(\hat{n}\,.\,\chi)] &= n\left(1-\frac{1}{\delta.}\right)\left(1-\frac{1}{\delta'.}\right)\left(1-\frac{1}{\delta''.}\right)\ldots.\frac{1}{\delta_1.}\,\frac{1}{\delta_1'.}\,\frac{1}{\delta_1''.}\ldots., \\ \overset{n}{\underset{1}{T}}[\breve{\bar{\mathsf{O}}}(\hat{n}\,.\,\kappa)\,.\,\mathsf{O}(\hat{n}\,.\,\chi)]^k &= n^k\left(1-\frac{1}{\delta.^k}\right)\left(1-\frac{1}{\delta'.^k}\right)\left(1-\frac{1}{\delta''.^k}\right)\ldots.\frac{1}{\delta_1^k.}\,\frac{1}{\delta_1'.^k}\,\frac{1}{\delta_1''.^k}\ldots., \end{aligned} \tag{78}$$

where $\hat{n}$ does not necessarily denote the same divisor of n under the sign O as under the sign $\breve{\bar{\mathsf{O}}}$.*

The condition κ in (77) and (78) is that which expresses any *bonâ fide* property or properties of the divisor, e. g. that the divisor be the sum of two squares, or that it be $\equiv 2$ (mod. 5). It cannot in general be a property of the quotient of n by the divisor. However, if ψ be a given property, there will, in general, be a condition κ to which, if d be subjected, the quotient of n by d will have the property ψ, as has been assumed above in (51) and (53). The following is an example of this mutual implication: the number of pairs of numbers ν, neither greater than n, and not *both* containing any divisor d of n, such that the quotient $\frac{n}{d}$ is of the form r_3, may be denoted by $\overset{n}{\underset{1}{T}}\left[\breve{\bar{\mathsf{O}}}(d=\hat{n})\,.\left(\frac{n}{d}=r_3\right)\right]^2$, and is given by the following table:—

* It may be remarked that, provided all the *least* divisors for the various conditions are used in (76), (77), and (78), the presence of such divisors in them, satisfying the imposed conditions, as are not included in the systems of least divisors does not affect the results.

Form of n.	r.	Conditions for ν.	$\overset{n}{\underset{1}{T}}\left[\breve{\mathrm{c}}\,(d=\hat{n})\,.\left(\frac{n}{d}=r_3\right)\right]^2$.
$3i\pm1$	±1	$\mathrm{c}\,\hat{n}\,.\,1_3$	$\overset{n}{\underset{1}{T}}[\breve{\mathrm{c}}\,(\hat{n}\,.\,1_3)]^2$
	∓1	$\mathrm{c}\,\hat{n}\,.\,2_3$	$\overset{n}{\underset{1}{T}}[\breve{\mathrm{c}}\,(\hat{n}\,.\,2_3)]^2$
	0	No such ν	n^2
$3^g(3i\pm1)$	±1	$\mathrm{c}\,3^g$	$\overset{n}{\underset{1}{T}}[\breve{\mathrm{c}}\,3^g]^2$
	∓1	$\mathrm{c}\,3^g\,.\,(\hat{n}\,.\,2_3)$	$\overset{n}{\underset{1}{T}}[\breve{\mathrm{c}}\,3^g\,.\,(\hat{n}\,.\,2_3)]^2$
$3^g(3i\pm1)$ $g>1$	0	$\mathrm{c}\,\hat{n}$	$\tau^2(n)$
$3^g(3i\pm1)$ $g=1$	0	$\mathrm{c}\,(\hat{n}\,.\,\bar{0}_3)$	$\overset{n}{\underset{1}{T}}[\breve{\mathrm{c}}\,(\hat{n}\,.\,\bar{0}_3)]^2$

(79)

where the upper or lower sign is to be used throughout any line in which the double sign occurs. Another example is given in (55).

Let λ, μ, ν, ... be any numerical quantities, positive or negative, and ϵ_1, ϵ_2, ϵ_3, ..., any succession of the signs $+$ and $-$ (or of the quantities $+1$ and -1); and let

$$\sigma=\epsilon_1\lambda+\epsilon_2\mu+\epsilon_3\nu+\ldots\,,\ \text{and}\ [f(\ )]\,\sigma=\epsilon_1 f(\lambda)+\epsilon_2 f(\mu)+\epsilon_3 f(\nu)+\ldots; \tag{80}$$

then I call $[f(\ )]\,\sigma$ *a functional distribute.* What is essential to the definition of a functional distribute is the manner in which σ is made up of terms λ, μ, ν, ..., and the sign prefixed to each; i. e. the *form of expression* of σ is essential. In the cases which I have occasion to consider here the expression σ is the complete development of an integral algebraic expression, the terms λ, μ, ν, ... being taken all positive, and therefore ϵ_1, ϵ_2, ϵ_3, ..., are the signs actually occurring in the development. For the sake of brevity I write, in these cases, σ in its undeveloped form. Thus, $\tau(n)$ being assumed of the form given in (76),

$$[f(\ )]\,\tau(n)=f(n)-f\left(\frac{n}{a}\right)-f\left(\frac{n}{b}\right)-f\left(\frac{n}{c}\right)-\ldots+f\left(\frac{n}{ab}\right)+f\left(\frac{n}{ac}\right)+f\left(\frac{n}{bc}\right)$$
$$+\ldots-f\left(\frac{n}{abc}\right)-\ldots, \tag{81}$$

$$[f(\ )]\left(\frac{1-a^{\alpha+1}}{1-a}\right)\left(\frac{1-b^{\beta+1}}{1-b}\right)\left(\frac{1-c^{\gamma+1}}{1-c}\right)\ldots\ldots=[f(\ )]\,(1+a+a^2+\ldots\ldots$$
$$+\,a^{\alpha})\,(1+b+b^2+\ldots\ldots+b^{\beta})\,(1+c+c^2+\ldots\ldots+c^{\gamma})\ldots\ldots$$
$$=f(1)+f(d)+f(d')+\ldots\ldots+f(n), \qquad (82)$$

where $a, b, c, \ldots\ldots, \alpha, \beta, \gamma, \ldots\ldots, d, d', \ldots\ldots$ are defined as above.

Professor Sylvester has called the function $[f(\ )]\tau(n)$ the *functional totient* of $f(n)$ and

$$[f(\ )]\left(\frac{1-a^{\alpha+1}}{1-a}\right)\left(\frac{1-b^{\beta+1}}{1-b}\right)\ldots\ldots$$

the *functional summant* of $f(n)$, and has denoted them by $(f\tau)n$ and $(f\sigma)n$, respectively (this Journal, Vol. II. pp. 386, 387).

List of Articles relating to the Subject of the foregoing Paper.

Cayley, A.: On Residuation in regard to a Cubic Curve. Messenger of Mathematics, 2d ser., III. pp. 62 – 65.

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Postscript to Note on a Point in Vulgar Fractions.

By J. J. Sylvester.

Let ϕx represent $x^2 - x + 1$, $\phi^n c$ will then be the general term of the "limiting sorites" whose first term is c, for which, if we please, $1 - c$ may be substituted. The properties of the numbers $\phi^n c$ seem to be worthy of some attention. I confine my observations in what follows to the lowest of such series, viz. where $c = 2$ or $- 1$.

The first five terms in such series then become $\bar{1}$ or 2, 3, 7, 43, 1807, 3263443, of which all but 1807, which $= 13.139$, are prime numbers. Every term in the series must contain only factors of the form $6i + 1$, and this, joined to the fact that a prime factor which has once appeared in any term can never reppear in any other, favors a tendency, so to say, of the numbers to remain primes, or at all events, to be of very limited frangibility into a product of primes.

It is easy to determine whether any proposed prime can occur as a factor of any term whatever in the series; for taking that number, say p, as a modulus, if r is a remainder of any term to that modulus, the remainder of the next term will be $r^2 - r + 1$, and as soon as any remainder reappears the series of remainders becomes periodic; so that necessarily in less than the number p of remainders, if p does divide any term of the sorites, we must arrive at a remainder zero, subsequent to which all the remainders are unity. I give the remainders and periods in the annexed table for all values of p of the form $6i + 1$ up to 139, from which it will be seen that, under that limit, 13 and 73 are the only prime numbers which are contained as factors in the terms of the series.

p	Remainders of $\phi^n(2)$ to modulus p.
2	0.
3	2, 0.
7	2, 3, 0.
13	2, 3, 7, 4, 0.
19	2, 3, 7, 5; 2, 3, 7, 5; ....
31	2, 3, 7, 12, 9, 11, 18, 28, 13; 2, 3, 7, ..., 13; ....
37	2, 3, 7; 6, 31; 6, 31; ....
43	2, 3, 7, 0.
61	2, 3, 7, 43, 38, 4, 13, 35, 32; 17, 29, 20, 15, 28, 25, 52, 30; ....
67	2, 3; 7, 43, 65; 7, 43, 65; ....
73	2, 3, 7, 43, 55, 51, 69, 21, 56, 15, 65, 0.
79	2, 3, 7; 43, 69, 32, 45, 6, 31, 61, 27, 71, 73; ....
97	2, 3, 7, 43, 61; 72, 69, 37; 72, 69, 37; ....
103	2, 3; 7, 43, 56, 94, 91, 54, 82, 51, 79, 86, 101; 7, 43, ...; ....
109	2, 3, 7, 43, 63, 92, 89, 94, 23, 71, 66, 40, 35, 101, 73, 25, 56, 29, 50, 53; 32, 12, 24, 8; 32, 12, 24, 8; ....
127	2, 3, 7, 43, 29, 51, 11; 111, 19, 89, 86, 72, 33, 41, 117; ....
139	2, 3, 7, 43, 0.
151	2, 3, 7, 43, 146; 31, 25, 148, 13, 6; ....
157	2, 3; 7, 43, 80, 41, 71, 104, 37, 77, 44, 9, 73, 76, 49, 155; ....
163	2, 3; 7, 43, 14, 20, 55, 37, 29, 161; ....
181	2, 3, 7, 43, 178, 13, 157, 58, 49, 0.
193	2; 3, 7, 43, 70, 6, 31, 159, 33, 92, 74, 192; ....
199	2, 3; 7, 43, 16, 42, 131, 116, 8, 57, 9, 73, 83, 41, 49, 164, 67, 45, 190, 91, 32, 197; ....

Instantaneous Proof of a Theorem of Lagrange on the Divisors of the Form $Ax^2 + By^2 + Cz^2$, *with a Postscript on the Divisors of the Functions which multisect the Primitive Roots of Unity.*

By J. J. Sylvester.

If possible, let p be not a divisor of $x^2 + y^2 + 1$, and consequently not of the form $4i + 1$, since, if it were of that form, $x^2 + 1$ would contain it.

Let ρ be any primitive p^{th} root of unity.

Call $R = \Sigma\rho^{x^2}$, where x^2 means any one of the quadratic residues of and inferiors to p, and let the period conjugate to R be called R'.

Let R^2 be expanded as a sum of powers of ρ. Then, because p is not of the form $4i + 1$, we cannot have $x^2 + y^2 = p$, so that no p^{th} power of ρ can occur in that expansion; again, because by hypothesis neither $2x^2$ nor $x^2 + y^2$ can be congruous to -1 [mod. p], no such power as ρ^{p-1} which belongs to R', nor consequently any other term of R', can appear in R^2; and as each power of ρ in R^2 belonging to the same period must appear a like number of times, we must have

$$R^2 = \frac{p-1}{2} R, \quad \text{i. e. } R = 0, \quad \text{or} \quad R = \frac{p-1}{2},$$

each of which suppositions is in the highest degree absurd. Hence p is a divisor of $x^2 + y^2 + 1$. Q. E. D.

Compare Legendre's *Théorie des Nombres*, Ed. 1830, Tom. 1, pp. 211–213, and again Serret's *Cours d'Algèbre Supérieure*, Tom. 2, pp. 94–99, for proofs of the more general similar theorem due to Lagrange, concerning $u^2 + Bt^2 + C$. These proofs are highly ingenious, but long and labored in no slight degree; and as the sole apparent object of either author in proving the general theorem is to make use of the particular case of it to which this note refers as a foundation to the proof of Fermat's law of the four squares, I have thought that an intuitive

proof of so important a lemma might not be without interest to some of the junior readers of the Journal.*

But in fact the general theorem may be proved with scarcely any greater trouble than the particular case disposed of.

For, supposing A, B, C to be all quadratic residues to p, we may write

$$A \equiv \alpha^2, \quad B \equiv \beta^2, \quad C \equiv \gamma^2 \text{ [mod. } p],$$
$$\alpha x = u, \quad \beta y = v, \quad \gamma z = w;$$

and the congruence $u^2 + v^2 + w^2 \equiv 0$, as previously shown, being soluble, evidently

$$Ax^2 + By^2 + Cz^2 \equiv 0$$

will be so too, since

$$\alpha x \equiv u, \quad \beta y \equiv v, \quad \gamma z \equiv w \text{ [mod. } p],$$

give integer values for x, y, z; and as obviously the case of A, B, C being all non-residues falls into the previous case by multiplying the congruence by any non-residue, we have only to consider the case of two of the three coefficients being residues and the third a non-residue, or the converse case, which, however, by multiplication as above, may be reduced to the former one.

Suppose, then, $A = \alpha^2$, $B = \beta^2$, C a non-residue, and that

$$Ax^2 + By^2 + Cz^2 \equiv 0 \text{ [mod. } p]$$

is insoluble. For simplicity, let $z = 1$. Then $u^2 + v^2 + C = 0$ must be insoluble; if p is of the form $4i + 3$, we shall obtain, precisely as before,

$$R^2 = \frac{p-1}{2} R,$$

and if p is of the form $4i + 1$,

$$R^2 = 2\frac{p-1}{4} + \left\{\left(\frac{p-1}{2}\right)^2 - \left(\frac{p-1}{2}\right)\right\} \div \frac{p-1}{2} \cdot R,$$

or $$R^2 - \frac{p-3}{2} R + \frac{p-1}{2} = 0, \quad \text{i.e.} \quad R = \frac{p-1}{2}, \quad \text{or} \quad R = -1,$$

any of which conclusions are eminently absurd.

---

* From this lemma there is scarcely more than a step to the theorem in question. If P is contained as a factor in the sum of four squares, it is easy to see that we may write $PQ = f^2 + g^2 + h^2 + k^2$, where $Q < P$, and

$$QQ' = (f - \alpha Q)^2 + (g - \beta Q)^2 + (h - \gamma Q)^2 + (k - \delta Q)^2,$$

where $Q' < Q$, and consequently, applying the Quaternion law of multiplication, $PQ' = f'^2 + g'^2 + h'^2 + k'^2$, and so we may form a continually decreasing series of quantities $Q, Q', Q'', \ldots$ any one of which multiplied by P is a sum of four squares. Hence any divisor of such sum is itself such a sum, but by the lemma any prime number is a divisor of the sum of three, which plays the same part for present purposes as a sum of four squares, and is therefore a sum of four squares; consequently any number whatever, by the rule of multiplication already alluded to in this note, will be a sum of four squares.

Hence $Ax^2 + By^2 + Cz^2 \equiv 0$ [mod. p] *cannot* be insoluble; i. e. the left-hand side of the congruence must contain p as a divisor.

P. S. In a future communication I will prove very simply that if a prime number $p = ef + 1$, and e is itself a prime number such that $(e - 1)$ contains no odd square number, then every divisor, without exception, (other than p) of the function whose roots are the e periods of the primitive p^{th} roots of unity, must be an e^{th} power residue of p. If $(e - 1)$ contains any square number, the proof still holds good, except as regards the factors of such square, and there is no reason at present for supposing that the theorem may not be extended to the case of these excepted factors.* The same kind of reasoning may be applied also to the theory of period-functions for which e (the number of the periods) is not a prime number, and I find for the case of $e = 4$, that, leaving out of account the number 2 (which is always a divisor of the four-period function to p when p is of the form $8i + 1$, but never when it is of the form $8i + 5$, and may be or not a biquadratic residue of p, according to a well-known law), the divisors of the four-period function (excepting p) which do not divide g (the even term in the equation $[f^2 + g^2 = p]$), are necessarily biquadratic residues of p; as is also true of the prime-number divisors of g which are of the form $4i + 1$; but the prime-number divisors of g (all of which are necessarily divisors of the four-period function), of the form $4i - 1$, are quadratic only, and not biquadratic residues of p when p is of the form $8i + 5$; whereas for the case of $p = 8i + 1$ *all* the odd divisors of the four-period function (not counting p) are biquadratic residues of p.† The same investigation leads to the remarkable conclusion that if $p = f^2 + 4\gamma^2$, where f and γ are both of them odd and p a prime number, every divisor of $\frac{f^2 + 3\gamma^2}{4}$ is a biquadratic residue of p, — a theorem which I imagine would be difficult to prove by any other method.

---

* Thus ex. gr., if e is a prime number of the form $2^{2^x} + 1$, I am able to prove that *every* divisor of the e-period function (not excepting 2, if 2 should happen to be such a divisor) is an e^{th}-power residue of p. Thus for $e = 2, 3, 5, 7, 11, 17$ we may be certain that there are none but e^{th}-power-residue divisors of the period-function.

† Of course in a certain sense p or zero is an any-power residue. But there is good reason for separating p from the residues proper, inasmuch as only the *first* power of p, but an *unlimited* power of any true e^{th}-power residue is a divisor of the e-period function, — a most important fact, which I presume must have been known to Bachmann, but has not been stated by him (in his *Kreistheilung*, 1872). An exceedingly simple proof of this and of the corresponding theorem for any cyclotomic function was given by Mr. Hathaway at a recent meeting of the Mathematical Seminarium, at the Johns Hopkins University.

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